Linear algebra

Vectors

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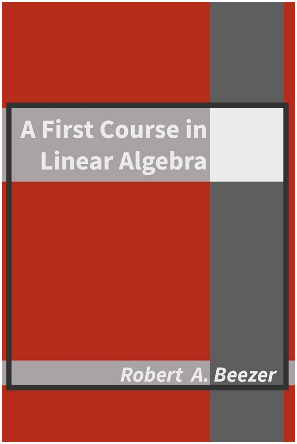
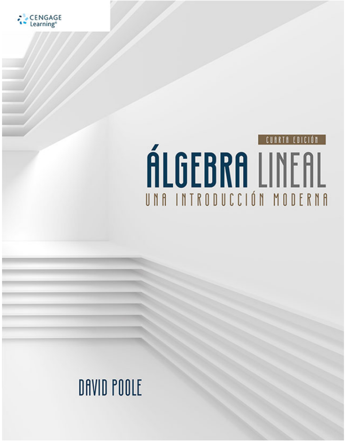
CONAHCYT
INAOE

July 9 2024

Contents

Bibliography





http://linear.ups.edu/

Vectors

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Physics

- Usually represented by arrows that have:
  - magnitude
  - and direction

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- List of numbers.

Vectors

Physics

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Mathematics

- Anything.
- As long as it respects certain rules.

Computer science

$$= \begin{bmatrix} 200{,}000 \\ 4 \\ 60 \end{bmatrix}$$

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Vectors in the plane

Definition

A **vector** is a directed line segment that corresponds to a displacement from one point A to another point B. The vector from A to B is denoted by $\overrightarrow{AB}$; the point A is called its **initial point**, or **tail**, and the point B is called its **terminal point** or **head**. Often, a vector is simply denoted by a single boldface, lowercase letters such as $\mathbf{v}$.

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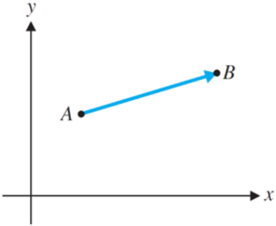


Vectors in the plane

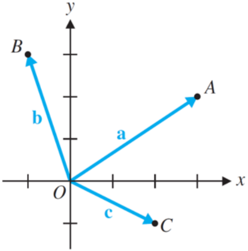
The set of all points in the plane corresponds to the set of all vector whose tail are at the origin O.

Definition

Vectors with its tail at the origin are called **position vectors**.

Vectors in the plane

Point A corresponds to the position vector $\mathbf{a} = \overrightarrow{OA} = [3, 2]$. The other vectors in the figure are $\mathbf{b} = [-1, 3]$ and $\mathbf{c} = [2, -1]$.



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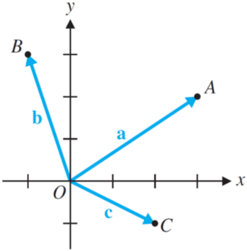


The individual coordinates (3 and 2 in the case of **a**) are called the **components** of the vector.

Vectors in the plane

Two vectors are equal if and only if their corresponding components are equal. Thus, $[x, y] = [1, 5]$ implies that $x = 1$ and $y = 5$.

Using **column vectors** instead of **row vectors** is frequently convenient.

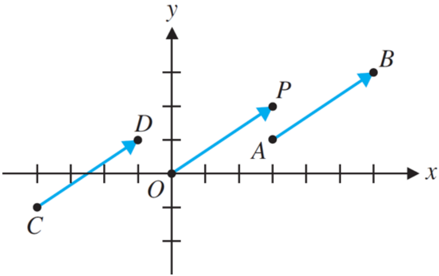
So, $[3, 2]$ can be represented as $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Vectors in the plane

We cannot draw the vector $[0,0] = \overrightarrow{OO}$ from the origin to itself. Nevertheless, it is a perfectly good vector and has a special name: the **zero vector**. The zero vector is denoted by $\mathbf{0}$.

Vectors in the plane

What can you say about these three vectors?



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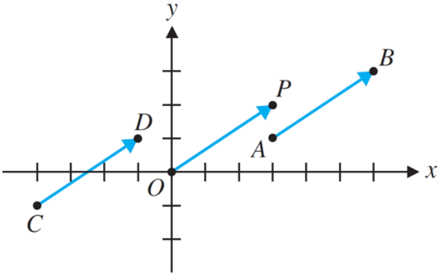
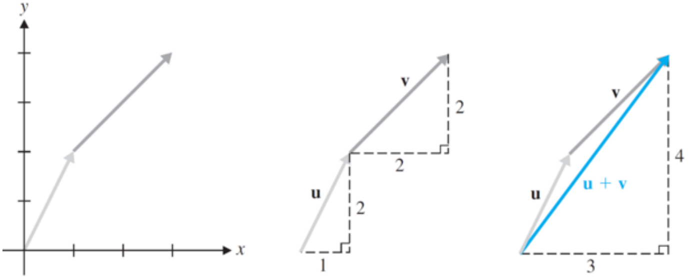


By setting the tail of each vector in the origin, we observe they all coincide.

New vectors from old

We often want to follow one vector by another. This leads to the notion of **vector addition**.

If we follow $\mathbf{u}$ by $\mathbf{v}$, we can visualize the total displacement as a third vector, denoted by $\mathbf{u} + \mathbf{v}$.



New vectors from old

In general, if $\mathbf{u} = [u_1, u_2]$ and $\mathbf{v} = [v_1, v_2]$, the their **sum** $\mathbf{u} + \mathbf{v}$ is the vector

$$\mathbf{u} + \mathbf{v} = [u_1 + v_1, u_2 + v_2]$$

New vectors from old

Our next vector operation is **scalar multiplication**. Given a vector **v** and a real number c, the **scalar multiplication** $c\mathbf{v}$ is the vector contained by multiplying each component of **v** by c. In general,

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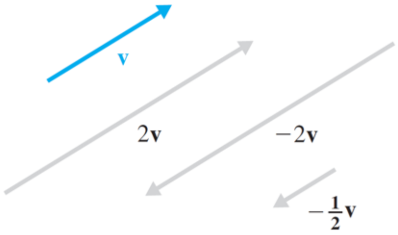
Geometrically, $c\mathbf{v}$ is a "scaled" version of $\mathbf{v}$.



Vectors in $\mathbb{R}^n$

$\mathbb{R}^n$ is a shorthand for $\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$, the cartesian product of $\mathbb{R}$ with itself n times. So, it is the set of all ordered n-tuples of real numbers written as row or column vectors. Thus, a vector **v** $\in \mathbb{R}^n$ is of the form

$$[v_1, v_2, .., v_n] \quad \text{or} \quad \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

The individual entries of **v** are its components; v_i is called the i-th component.

Vectors in $\mathbb{R}^n$

We extend the definitions of vector addition and scalar multiplication to $\mathbb{R}^n$ in the obvious way:

If $\mathbf{u} = [u_1, u_2, ..., u_n]$ and $\mathbf{v} = [v_1, v_2, ..., v_n]$, the i-th component of $\mathbf{u} + \mathbf{v}$ is $u_i + v_i$ and the i-th component of $c\mathbf{v}$ is just cv_i.

Vectors in $\mathbb{R}^n$

Algebraic properties of vectors in $\mathbb{R}^n$.

Theorem

Let $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *be vectors in* $\mathbb{R}^n$ *and let* c *and* d *be scalars. Then*

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- $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ *(additive associativity)*
- $\mathbf{u}+\mathbf{0}=\mathbf{u}$ *(zero vector)*
- $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$ *(additive inverses)*
- $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$ *(distributivity across vector addition)*
- $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$ *(distributivity across scalar addition)*
- $c(d\mathbf{u})=(cd)\mathbf{u}$ *(scalar multiplication associativity)*
- $1\mathbf{u}=\mathbf{u}$ *(one)*

Vectors in $\mathbb{R}^n$

Each bullet must be proved. In general, they all inherit the properties of the operations over real numbers. For instance,

$$\mathbf{u}+\mathbf{v}=\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}+\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}=\begin{bmatrix}u_1+v_1\\u_2+v_2\\\vdots\\u_n+v_n\end{bmatrix}=\begin{bmatrix}v_1+u_1\\v_2+u_2\\\vdots\\v_n+u_n\end{bmatrix}=\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}+\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}=\mathbf{v}+\mathbf{u}$$

Example

Simplify ($\mathbf{x}$ in terms of $\mathbf{a}$)

$$5\mathbf{x} - \mathbf{a} = 2(\mathbf{a} + 2\mathbf{x})$$

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$$5\mathbf{x} - \mathbf{a} = 2(\mathbf{a} + 2\mathbf{x})$$
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(-\mathbf{a} + 5\mathbf{x}) - 4\mathbf{x} &= 2\mathbf{a} + (4\mathbf{x} - 4\mathbf{x}) \\
-\mathbf{a} + (5\mathbf{x} - 4\mathbf{x}) &= 2\mathbf{a} + \mathbf{0}
\end{aligned}
$$

$$
\begin{aligned}
-\mathbf{a} + (5 - 4)\mathbf{x} &= 2\mathbf{a} \\
-\mathbf{a} + (1)\mathbf{x} &= 2\mathbf{a} \\
-\mathbf{a} + \mathbf{x} &= 2\mathbf{a} \\
\mathbf{a} + (-\mathbf{a} + \mathbf{x}) &= \mathbf{a} + 2\mathbf{a} \\
(\mathbf{a} + (-\mathbf{a})) + \mathbf{x} &= (1 + 2)\mathbf{a} \\
\mathbf{0} + \mathbf{x} &= 3\mathbf{a} \\
\mathbf{x} &= 3\mathbf{a}
\end{aligned}
$$

Linear combinations and coordinates

Linear combinations and coordinates

Definition

A vector $\mathbf{v}$ is a **linear combination** of vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k$ if there are scalars $c_1, c_2, ..., c_k$ such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

The scalars $c_1, c_2, ..., c_k$ are called the **coefficients** of the linear combination.

Example

Let $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. We can use **u** and **v** to locate a new set of axes (in the same way that $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ locate the standard coordinate axes). We can use these new axes to determine a **coordinate grid** that will let us easily locate linear combinations of **u** and **v**.

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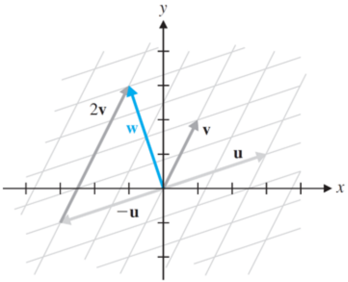

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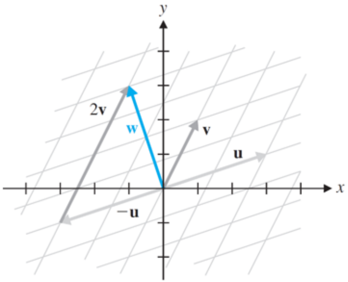

$$\mathbf{w} = -\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

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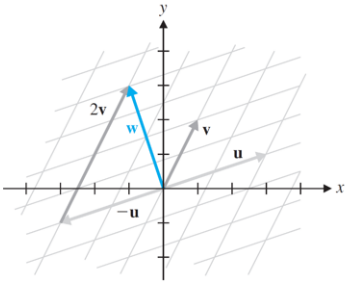

$$\mathbf{w} = -\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

(Observe that -1 and 3 are the coordinates of $\mathbf{w}$ with respect to $\mathbf{e}_1$ and $\mathbf{e}_2$.)

The dot product

The dot product

The vector versions of length, distance, and angle can all be described using the notion of the dot product of two vectors.

The dot product

Definition

If

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

then the **dot product** $\mathbf{u} \cdot \mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots u_n v_n$$

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$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots u_n v_n$$

Since $\mathbf{u} \cdot \mathbf{v}$ is a number, it is sometimes called the **scalar product** of $\mathbf{u}$ and $\mathbf{v}$.

The dot product

Theorem

Let $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *be vectors in* $\mathbb{R}^n$ *and let* c *be a scalar. Then*

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ *(commutativity)*
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w}$ *(distributivity)*
- $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$
- $\mathbf{u} \cdot \mathbf{u} \geq 0$
- $\mathbf{u} \cdot \mathbf{u} = 0$ *if and only if* $\mathbf{u} = \mathbf{0}$

The dot product

Each bullet must be proved. For instance,

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} &= u_1v_1 + u_2v_2 + \cdots + u_nv_n \\ &= v_1u_1 + v_2u_2 + \cdots + v_nu_n = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \mathbf{v}\cdot\mathbf{u}\end{aligned}$$

Example

Show that $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v}$

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$$
\begin{aligned}
(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) &= (\mathbf{u}+\mathbf{v})\cdot\mathbf{u} + (\mathbf{u}+\mathbf{v})\cdot\mathbf{v} \\
&= \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} \\
&= \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} \\
&= \mathbf{u}\cdot\mathbf{u} + 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v}
\end{aligned}
$$

Length

The **length** (or **norm**) of a vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ is the nonnegative scalar defined by

$$||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

Length

Theorem

Let $\mathbf{v}$ *be a vector in* $\mathbb{R}^n$ *and let* c *be a scalar. Then*

- $||\mathbf{v}|| = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$
- $||c\mathbf{v}|| = |c|\ ||\mathbf{v}||$

Length

Theorem

Let $\mathbf{v}$ *be a vector in* $\mathbb{R}^n$ *and let* c *be a scalar. Then*

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Proof.

(b)

$$\begin{aligned} ||c\mathbf{v}||^2 = c\mathbf{v}\cdot c\mathbf{v} &= c^2v_1^2 + c^2v_2^2 + \cdots + c^2v_n^2 \\ &= c^2(v_1^2 + v_2^2 + \cdots + v_n^2) \\ &= c^2(\mathbf{v}\cdot\mathbf{v}) = c^2||\mathbf{v}||^2 \end{aligned}$$

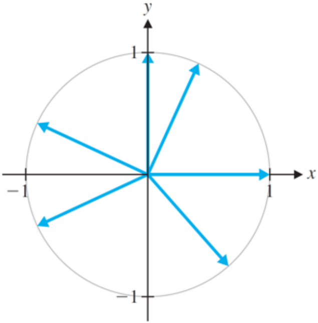
Apply the square root function in both sides

$$||c\mathbf{v}|| = |c|\ ||\mathbf{v}||$$

$\square$

Length

A vector of length 1 is called a **unit vector**. In $\mathbb{R}^2$, the set of all unit vectors can be identified with the unit circle, the circle of radius 1 centered at the origin.



Length

Given any nonzero vector $\mathbf{v}$, we can always find a unit vector in the same direction as $\mathbf{v}$ by dividing $\mathbf{v}$ by its own length (or, equivalently, multiplying by $1/||\mathbf{v}||$).

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$$\begin{aligned} ||\mathbf{u}|| &= ||(1/||\mathbf{v}||)\mathbf{v}|| \\ &= |\ 1/||\mathbf{v}||\ |\ ||\mathbf{v}|| \\ &= (1/||\mathbf{v}||)||\mathbf{v}|| \\ &= 1 \end{aligned}$$

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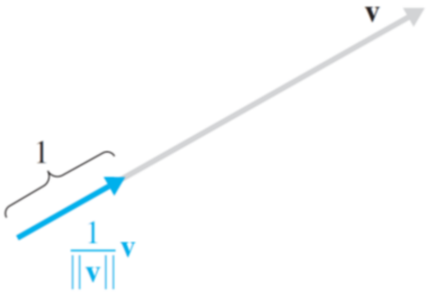
and $\mathbf{u}$ is in the same direction as $\mathbf{v}$, since $1/||\mathbf{v}||$ is a positive scalar.

Length

Finding a unit vector in the same direction is often referred to as **normalizing** a vector.



Length

In general, in $\mathbb{R}^n$, we define unit vectors $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n$, where $\mathbf{e}_i$ has 1 in its i-th component and zeros elsewhere.

These vectors arise repeatedly in linear algebra and are called the **standard unit vectors**.

Example

Normalize the vector $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

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So, the unit vector in the same direction as $\mathbf{v}$ is given by

$$\mathbf{u} = \frac{1}{||\mathbf{v}||}\mathbf{v} = \frac{1}{\sqrt{14}} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{14} \\ -1/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}$$

Length

Theorem

The Cauchy-Schwarz inequality. For all vectors **u** *and* **v** *in* $\mathbb{R}^n$

$$|\mathbf{u} \cdot \mathbf{v}| \leq ||\mathbf{u}|| \; ||\mathbf{v}||$$

Proof.

This inequality is equivalent to

$$(\mathbf{u} \cdot \mathbf{v})^2 \leq ||\mathbf{u}||^2 \; ||\mathbf{v}||^2$$

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In $\mathbb{R}^2$, $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

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$$(u_1 v_1 + u_2 v_2)^2 \leq^? (u_1^2 + u_2^2)(v_1^2 + v_2^2)$$
$$u_1^2 v_1^2 + 2u_1 v_1 u_2 v_2 + u_2^2 v_2^2 \leq^? u_1^2 v_1^2 + u_1^2 v_2^2 + u_2^2 v_1^2 + u_2^2 v_2^2$$
$$2u_1 v_1 u_2 v_2 \leq^? u_1^2 v_2^2 + u_2^2 v_1^2$$

Length

Proof (cont.)

$$2u_1v_1u_2v_2 \leq^? u_1^2v_2^2 + u_2^2v_1^2$$
$$2(u_1v_2)(u_2v_1) \leq^? (u_1v_2)^2 + (u_2v_1)^2$$

Length

Proof (cont.)

$$2u_1v_1u_2v_2 \leq^? u_1^2v_2^2 + u_2^2v_1^2$$
$$2(u_1v_2)(u_2v_1) \leq^? (u_1v_2)^2 + (u_2v_1)^2$$

Let $a = u_1v_2$ and $b = u_2v_1$

$$2ab \leq^? a^2 + b^2$$
$$0 \leq^? a^2 + b^2 - 2ab$$

Length

Proof (cont.)

$$2u_1v_1u_2v_2 \leq^? u_1^2v_2^2 + u_2^2v_1^2$$
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Since

$$a^2 + b^2 - 2ab = (a-b)^2 \geq 0$$

Length

Proof (cont.)

$$2u_1v_1u_2v_2 \leq^? u_1^2v_2^2 + u_2^2v_1^2$$
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Since

$$a^2 + b^2 - 2ab = (a-b)^2 \geq 0$$

we can remove the "?" sign from all the previous inequalities. (In a conventional style, the proof goes backward). □

Length

Theorem

The triangle inequality. For all vectors $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^n$

$$||\mathbf{u}+\mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$$

Proof.

$$\begin{aligned} ||\mathbf{u}+\mathbf{v}||^2 &= (u_1+v_1)^2 + \cdots + (u_n+v_n)^2 \\ &= (\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}+\mathbf{v}) \\ &= \mathbf{u}\cdot\mathbf{u} + 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v} \\ &\leq ||\mathbf{u}||^2 + 2|\mathbf{u}\cdot\mathbf{v}| + ||\mathbf{v}||^2 \\ &\leq ||\mathbf{u}||^2 + 2||\mathbf{u}||\ ||\mathbf{v}|| + ||\mathbf{v}||^2 \\ &= (||\mathbf{u}|| + ||\mathbf{v}||)^2 \end{aligned}$$

Length

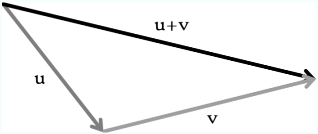
Theorem

The triangle inequality. For all vectors $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^n$

$$||\mathbf{u}+\mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$$

Proof.

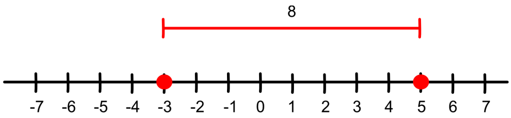
$$\begin{aligned}
||\mathbf{u}+\mathbf{v}||^2 &= (u_1+v_1)^2 + \cdots + (u_n+v_n)^2 \\
&= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) \\
&= \mathbf{u}\cdot\mathbf{u} + 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v} \\
&\leq ||\mathbf{u}||^2 + 2|\mathbf{u}\cdot\mathbf{v}| + ||\mathbf{v}||^2 \\
&\leq ||\mathbf{u}||^2 + 2||\mathbf{u}||\ ||\mathbf{v}|| + ||\mathbf{v}||^2 \\
&= (||\mathbf{u}|| + ||\mathbf{v}||)^2
\end{aligned}$$



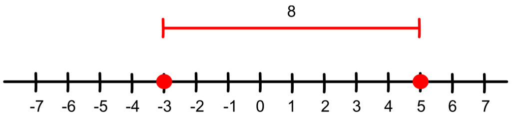
$\square$

Distance

Distance



Distance



$$d(5,-3) = |5-(-3)| = |-3-5|$$

Distance

Definition

The **distance** $d(\mathbf{u}, \mathbf{v})$ between vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is defined by

$$d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$$

Distance

Definition

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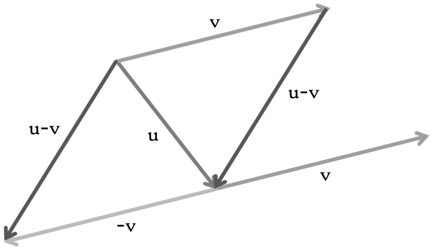

Example

Find the distance between $\mathbf{u} = \begin{bmatrix} \sqrt{2} \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$

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Find the distance between $\mathbf{u} = \begin{bmatrix} \sqrt{2} \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$

$$\mathbf{u} - \mathbf{v} = \begin{bmatrix} \sqrt{2} \\ -1 \\ 1 \end{bmatrix}$$

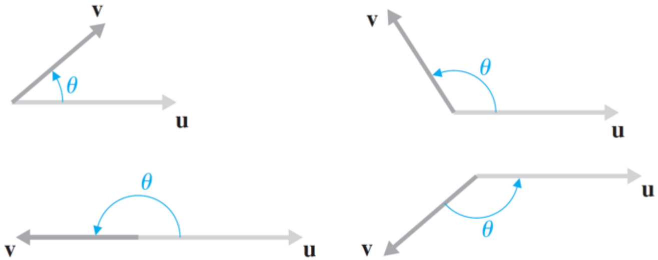
So,

$$d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}|| = \sqrt{(\sqrt{2})^2 + (-1)^2 + 1^2} = \sqrt{4} = 2$$

Angles

Angles

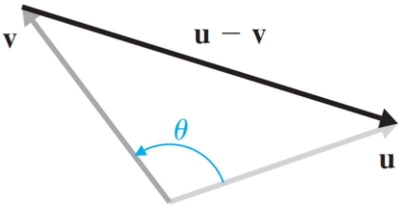
The dot product can also be used to calculate the angle between a pair of vectors. In $\mathbb{R}^2$ or $\mathbb{R}^3$, the angle between the nonzero vector **u** and **v** will refer to the angle θ determined by these vectors that satisfies $0 \leq \theta \leq 180$.



Angles

Consider the triangle with sides $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}-\mathbf{v}$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. Applying the law of cosines to this triangle yields

$$||\mathbf{u}-\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2||\mathbf{u}||\ ||\mathbf{v}|| \cos\theta$$



Angles

After simplification, we get

$$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}|| \; ||\mathbf{v}|| \cos \theta$$

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Definition

For nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$,

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||\ ||\mathbf{v}||}$$

Angles

After simplification, we get

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Definition

For nonzero vectors **u** and **v** in $\mathbb{R}^n$,

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||\ ||\mathbf{v}||}$$

By Cauchy-Schwarz $\left|\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\ ||\mathbf{v}||}\right| \leq 1$. So, $\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\ ||\mathbf{v}||}$ take values between -1 and 1.

Orthogonal vectors

We now generalize the idea of perpendicularity to vectors in $\mathbb{R}^n$, where it is called **orthogonality**.

In $\mathbb{R}^2$ or $\mathbb{R}^3$, two nonzero vectors **u** and **v** are **perpendicular** if the angle θ between them is a right angle - that is, if $\theta = \pi/2$ radians, or 90.

Orthogonal vectors

Thus,

$$\frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| \ ||\mathbf{v}||} = \cos 90 = 0$$

and it follows that $\mathbf{u} \cdot \mathbf{v} = 0$. This motivates the following definition.

Orthogonal vectors

Thus,

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Definition

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are **orthogonal** to each other if $\mathbf{u} \cdot \mathbf{v} = 0$.

Orthogonal vectors

Thus,

$$\frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| \; ||\mathbf{v}||} = \cos 90 = 0$$

and it follows that $\mathbf{u} \cdot \mathbf{v} = 0$. This motivates the following definition.

Definition

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are **orthogonal** to each other if $\mathbf{u} \cdot \mathbf{v} = 0$.

Since $\mathbf{0} \cdot \mathbf{v}$ for every vector in $\mathbb{R}^n$, the zero vector is orthogonal to every vector.

Orthogonal vectors

Thus,

$$\frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| \ ||\mathbf{v}||} = \cos 90 = 0$$

and it follows that $\mathbf{u} \cdot \mathbf{v} = 0$. This motivates the following definition.

Definition

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are **orthogonal** to each other if $\mathbf{u} \cdot \mathbf{v} = 0$.

Since $\mathbf{0} \cdot \mathbf{v}$ for every vector in $\mathbb{R}^n$, the zero vector is orthogonal to every vector.

Is the zero vector orthogonal to itself?

Orthogonal vectors

Theorem

Pythagora's theorem. For all vectors $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^n$

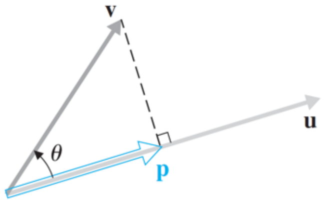
$$||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$$

if and only if $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal.*

Projections

Projections

Consider two nonzero vectors **u** and **v**. Let **p** be the vector obtained by dropping a perpendicular from the head of **v** onto **u** and let θ be the angle between **u** and **v**.



Projections

Definition

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$ and $\mathbf{u} \neq \mathbf{0}$, the **projection of v onto u** is the vector $proj_{\mathbf{u}(\mathbf{v})}$ defined by

$$proj_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}\right)\mathbf{u}$$

(You can prove it for $\mathbb{R}^2$)

Ending

Homework

- You have three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ such that $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$. Is always $\mathbf{v} = \mathbf{w}$?
- Prove that $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||\ ||\mathbf{v}|| \cos\theta$ (slide 46).
- Prove the Pythagora's theorem for vectors in $\mathbb{R}^n$ (slide 49).
- Prove the definition of projection over $\mathbb{R}^2$ (slide 52).
- $||proj_{\mathbf{u}}(\mathbf{v})|| \leq ||\mathbf{v}||$ in $\mathbb{R}^2$ and $\mathbb{R}^3$ (Can you **see** why?).
    - Show that this inequality is true in $\mathbb{R}^n$.
    - Show that this inequality is equivalent to the Cauchy-Schwarz inequality.

Next topics

A bit more on vectors

Thank you