Computing general companion matrices and stability regions of multirate methods

Gustavo Rodríguez–Gómez¹*, Pedro González–Casanova ², Jorge Martínez–Carballido ³

¹ Ciencias Computacionales, Instituto Nacional de Astrofísica, Óptica y Electrónica, Apdo. Postal 51 y 216, Puebla, Pue., C.P. 72000, México, e-mail: grodrig@inoaep.mx
² Cómputo Científico, DGSCA, Universidad Nacional Autónoma de México, Ciudad Universitaria, Circuito Exterior, C.P. 04510, México D.F., e-mail: pedrogc@plectics.dgsc2.unam.mx
³ Ciencias Computacionales, Instituto Nacional de Astrofísica, Óptica y Electrónica, Apdo. Postal 51 y 216, Puebla, Pue., C.P. 72000, México, e-mail:jmc@inoaep.mx

SUMMARY

We obtain the necessary and sufficient conditions that guarantee the absolute stability of semi–implicit multirate linear discretizations of 2 × 2 differential systems. This result is based on the construction of a general companion matrix that characterizes the class of multirate discretizations built from, linear and zero–order interpolation. These results are numerically verified for particular examples whose stability regions are graphically displayed.

1. Introduction

Simulation of large dynamic systems frequently leads to integrate systems of ordinary differential equations whose solution can be decomposed into a small number of fast components plus a set of slow components. Standard numerical integration methods failed to efficiently approximate the solution, since the fast components restrict the step size needed to integrate also the slow components. We should expect to gain an advantage if the fast solution can be integrated with a small step size, while the remaining components are integrated by using a relatively large step size.

Multirate integration methods are techniques that use different step sizes with different or equal integration schemes to approximate the fast and the slow components of the solution.

*Correspondence to: Gustavo Rodríguez–Gómez

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These methods are used in real-time simulation [12], modular dynamic systems simulation [9], electrical network simulation [6].

One of the major problems concerning the use of multirate methods to solve real-life problems is the lack of general theoretical results that guarantee their absolute stability [7]. However, although a general stability approach for these methods is still unknown, several results have appeared in the literature that analyzes the absolute stability conditions for different multirate methods.

The above results can be classified in two groups according to the underlying differential system to be integrated. In the first group, the stability conditions are derived for multirate methods used to integrate a $2 \times 2$ linear block systems of ordinary differential equations [4, 11, 13, 14]. In the second group, these conditions are determined for multirate methods that integrate a $2 \times 2$ linear scalar differential system [5, 6, 7, 8, 17]. The research developed within the second group, makes possible to take advantage of this simpler setting to go further into the formulation of a general stability approach.

In this article we aim to analyze the absolute stability conditions for the class of semi-implicit multirate linear methods based on full linear polynomial interpolation or zero-order interpolation, which in particular includes forward and backward information methods. This work is placed within linear scalar systems type, as the former second group.

In [4, 7, 11, 13, 14, 16, 17], the stability conditions of each multirate method were obtained by explicitly deriving the corresponding particular companion matrix. The characterization of these matrices is the starting point of the stability analysis, it generally involves a considerable number of algebraic manipulations. Thus, it would be desirable to advance toward a unification of these matrices under a common algebraic structure.

In this work we determine the explicit general structure of the companion matrix for the class of semi-implicit multirate linear methods. Each particular companion matrix, related to the multirate methods that belong to this class, by construction is included in the structure of this general matrix.

We also aim to use the above general characterization of the companion matrix to obtain the stability conditions for this class of multirate techniques.

The absolute stability conditions for different multirate methods are frequently deriving by applying the Routh–Hurwitz conditions to the corresponding characteristic polynomial [7, 8, 11]. This approach involves a large number of algebraic manipulations.

Our approach is a top-down process. First, we obtain the general companion matrix for a class of multirate method. Second, we get the particular companion matrix of the multirate method considered. Third, we derive the stability conditions from this matrix rather than from the characteristic polynomial. This approach considerably reduces the algebraic manipulations to perform the stability analysis.

Hence, a second goal of this article is to derive a simple set of stability conditions which are directly related to the companion matrix rather than to the characteristic polynomial.

Besides, we find, as opposed to what it could be expected, that the stability region of the multirate linear interpolation based method does not necessarily include the stability region of the zero-order interpolation scheme. These results are verified for particular examples whose stability regions are graphically displayed. Similar results have been reported for multirate Runge–Kutta methods in [7].

This paper is organized as follows, in Section two, we derive the general characterization of the companion matrix, and we obtain the necessary and sufficient conditions that guarantee
the absolute stability of the multirate linear methods. In Section three we describe a graphic method to display the stability regions by showing their boundaries in the plane. In this section, we also perform numerical experiments to find the absolute stability regions of the forward Euler multirate based on full linear interpolation or zero-order interpolation. Finally, in Section four conclusions are given.

2. Stability Analysis of Semi–Implicit Multirate Linear Methods

In this section we characterize the absolute stability conditions for the class of semi–implicit multirate linear methods. We first analyze these multirate methods for a multistep \((p \geq 1)\) discretization. Here, we obtain three block companion matrices corresponding to the multirate methods based on full linear interpolation or zero–order interpolation; see Theorem 2.2, below. These three matrices are then analyzed for the one–step \((p = 1)\) case, where a single \(2 \times 2\) companion matrix is obtained for the three previously mentioned cases; see Corollary 2.4, below.

These results are then used to provide an explicit characterization of the stability conditions. In the one–step discretization case we obtain, in Lemma 2.5, necessary and sufficient conditions that guarantee the absolute stability for the class of linear semi–implicit multirate methods.

Consider the initial value problem (IVP) defined by the differential equation system

\[
\begin{pmatrix}
y' \\
z'
\end{pmatrix} = \begin{pmatrix} f(x, y, z) \\
g(x, y, z) \end{pmatrix},
\]

with initial conditions \((x_0, y_0, z_0)^T\) at \(x = a\). The prime \((')\) means \(d/dx\), and \(f: \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^{N_1}\), \(g: \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^{N_2}\) for \(N = N_1 + N_2\).

For expository purposes, we discuss the case where \(N_1 = 1\), \(N_2 = 1\) and \(N = 2\). The variable \(x\) belongs to the interval \(I = [a, b]\), and we assume that \(y\) is the fast component and \(z\) is the slow component. We seek a solution in the range \(a \leq x \leq b\), and assume that \(f, g\) satisfies conditions which guarantee that the IVP (1) has unique continuously differentiable solution, that we indicate by \((y(x), z(x))^T\).

We wish to approximate the solution of the IVP (1) by a multirate linear multistep method. Let \(H = kh\) be the step size of \(z\), \(h\) the step size of \(y\), \(k\) is a positive integer that is called the multirate factor. The multirate linear multistep method for the numerical solution of the IVP (1) is given by

\[
y_{\sigma+1} = \sum_{j=0}^{p-1} \left( \alpha_j y_{\sigma-j} + h\beta_j^* \hat{f}_{\sigma-j} \right) + h\beta_{p*}^* \hat{f}_{\sigma+1},
\]

\[
z_{(n+1)k} = \sum_{j=0}^{p-1} \left( \alpha_j z_{(n-j)k} + H\beta_j g_{(n-j)k} \right) + H\beta_{p*} g_{(n+1)k},
\]

where \(\sigma = nk + i\) for \(i = 0, 1, \ldots, k - 1\), \(\hat{f}_\sigma = f(x_\sigma, y_\sigma, \hat{z}_\sigma)\), \(g_{nk} = g(x_{nk}, y_{nk}, z_{nk})\), and \(\hat{z}_\sigma\) is an approximation to \(z_\sigma\). Without loss of generality and to simplify notation, we can assume that both methods have \(p\)–steps.

The process of integrating the slow component over one step \(H\) is referred to as a macro step, and the process of integrating the fast component with a setp size \(h\) is referred to as...
**micro step.** A complete multirate step contains one macro step and \( k \) micro steps and it is called a compound step.

Note that if \( \beta_p = 0 \) in (3), the method is explicit for \( z \). This class of multirate formula is called semi-implicit \([10]\).

To approximate the values of \( z \) that are not in the mesh, we consider the following three strategies:

1. **Linear interpolation:** \( \hat{z}_\sigma(x) \) interpolates linearly to \( z_\sigma \).
2. **Forward information:** \( \hat{z}_\sigma(x) = z_{(n+1)k} \), for all \( x \).
3. **Backward information:** \( \hat{z}_\sigma(x) = z_{nk} \), for all \( x \).

The second case, forward information methods, are used in simulation problems \([18]\).

We need at least a two-dimensional linear test equation, to perform the stability analysis of multirate methods:

\[
\begin{pmatrix}
y' \\
z'
\end{pmatrix} = \begin{pmatrix}
a_{11} & \mu \\
\varepsilon & a_{22}
\end{pmatrix} \begin{pmatrix}
y \\
z
\end{pmatrix},
\]

where \( a_{11}, \mu, \varepsilon, a_{22} \) can be real or complex numbers, the fast component is \( y \), the slow component is \( z \), and their eigenvalues are \( \lambda_p, \lambda_s \) respectively.

To facilitate the analysis, for multirate linear equations (2) and (3), we use the terminology pointed out in Appendix. Furthermore, as usual, we denote by \( e_k \) the column vector with 1 in position \( k \) and 0 elsewhere.

In the slow component case, let \( \mathbf{a}_s = (a_0, a_1, \ldots, a_{p-1}, \beta_0, \beta_1, \ldots, \beta_p)^T \), \( \Phi_s = \Phi(\mathbf{a}_s) \),

\[
\Gamma_s = \beta_p \mathbf{e}_1 + \mathbf{e}_2,
\]

and \( Z_{(n+1)k} \) the \( 2p \times 1 \) matrix given by

\[
Z_{(n+1)k} = (z_{(n+1)k}, H g_{(n+1)k}, z_{nk}, H g_{nk}, \ldots, z_{(n-p+2)k}, H g_{(n-p+2)k})^T;
\]

see Appendix equations (35) to (38).

Similarly, for the fast component we have \( \mathbf{a}_f = (a_0^*, a_1^*, \ldots, a_{p-1}^*, \beta_0^*, \beta_1^*, \ldots, \beta_p^*)^T \), \( \Phi_f = \Phi(\mathbf{a}_f) \), \( \Gamma_f = \beta_p^* \mathbf{e}_1 + \mathbf{e}_2 \), and the \( 2p \times 1 \) matrix

\[
Y_{\sigma+1} = (y_{\sigma+1}, h \hat{f}_{\sigma+1}, y_\sigma, h \hat{f}_\sigma, \ldots, y_{\sigma-p+2}, h \hat{f}_{\sigma-p+2})^T.
\]

Therefore, the multirate linear method given by (2) and (3) can be written in the form

\[
\begin{align*}
Y_{\sigma+1} &= \Phi_f Y_\sigma + h \Gamma_f \hat{f}_{\sigma+1}, \\
Z_{(n+1)k} &= \Phi_s Z_{nk} + H \Gamma_s g_{(n+1)k}.
\end{align*}
\]

When \( \hat{z} \) is a linear polynomial, we need to express it in a compatible way relative to the notation introduced in Appendix. In this case, let \( x_{nk}, x_{(nk+1)k} \) be distinct points in \([a, b]\) and let \( \hat{z} \) be the polynomial of degree one interpolating to the corresponding function values \( z_{nk}, z_{(nk+1)k} \) of the slow component \( z \), that is

\[
\hat{z}_{\sigma+1} = z_{nk} + (i + 1)h g_{nk},
\]

where \( \sigma = nk + i \) for \( i = 0, 1, \ldots, k-1 \).

We define the \( 2p \times 1 \) matrix \( \hat{Z}_{\sigma+1} = (\hat{z}_{\sigma+1}, h g_{\sigma+1}, 0, \ldots, 0)^T \) and the \( 2p \times 2p \) matrix \( \Phi_i = ((\mathbf{e}_1 + ((1 + i)/k) \mathbf{e}_2)^T, 0, \ldots, 0)^T \). Hence, we can write \( \hat{Z}_{\sigma+1} \) as

\[
\hat{Z}_{\sigma+1} = \Phi_i Z_{nk} + h \mathbf{e}_2 g_{\sigma+1}.
\]
By substituting $g_{\sigma+1} = \varepsilon y_{\sigma+1} + a_{22} \hat{z}_{\sigma+1}$ into (8); see test equation (4), we can denote (7) in a compatible way relative to the notation introduced in Appendix, by

$$\hat{Z}_{\sigma+1} = M_i Z_{nk} + \frac{1}{k} \Theta_s Y_{\sigma+1},$$

where $\sigma = nk + i$ for $i = 0, 1, \ldots, k - 1$, and

$$M_i = \left( I + \frac{1}{k} \Theta e^T \right) \Phi_i, \quad \Theta_s = \varepsilon_2 e^T,$$

the matrix $I$ is the identity matrix.

Consequently, if the test equation (4) is solved by the multirate linear method given by (5) and (6), a compound step of the multirate scheme can be expressed as $\hat{Y}_{nk} = M \hat{Y}_{nk}$, where $\hat{Y}_{nk} = (Y_{nk}, Z_{nk})^T$, and $M$ is the companion matrix.

Therefore, to perform the absolute stability analysis of multirate linear formulas, we proceed in a similar way to the conventional linear multistep methods. The eigenvalues of the characteristic polynomial of the companion matrix $M$ must be into the unitary disk: $\rho(M) < 1$ where $\rho$ is the spectral radius [3].

**Definition 2.1.** The multirate linear multistep method (5), (6) is said to be absolutely stable for a particular problem, with micro step $h$ and multirate factor $k$, if $\rho(M) < 1$.

We are interested in knowing the class of particular problems for which the multirate linear method is absolutely stable, with micro step $h$ and multirate factor $k$.

Our task is now to determine the general structure of the companion matrix $M$, for the multirate linear multistep method based on full linear interpolation or zero–order interpolation.

**Theorem 2.2.** Suppose that we use a semi–implicit multirate linear method defined by Equations (5), (6) to solve the linear scalar test equation (4), and assume that full linear interpolation or zero–order interpolation is used to approximate the values of the slow variable $z$ that are not in the grid. Then, the general structure of the companion matrix $M$ is given by:

- **Linear interpolation**
  $$M = \begin{pmatrix} M_f^k & S_f \\ \Theta_s M_f^k & M_s + \Theta_s S_f \end{pmatrix}.$$  

- **Forward information**
  $$M = \begin{pmatrix} M_f^k & F_f \Theta_f M_s \\ \Theta_s M_f^k & (I + \Theta_s F_f \Theta_f) M_s \end{pmatrix}.$$  

- **Backward information**
  $$M = \begin{pmatrix} M_f^k & F_f \Theta_f \\ \Theta_s M_f^k & M_s + \Theta_s F_f \Theta_f \end{pmatrix},$$

where $M_s, \Theta_s$ as in (12), $M_f, \Theta_f$ as in (15), $I$ is the identity matrix, $S_f$ and $F_f$ as in (19) and (21) respectively.

**Proof.** The aim is to divide the proof in three cases: linear interpolation, forward and backward information for the fast component. The slow component has the same structure in all cases; however it depends on $Y_{(n+1)k}$; see (11) below.
Suppose that we use the semi-implicit multirate method (5), (6) to solve the test equation (4). From (4), it follows that
\[ g(n+1)_k = \varepsilon y(n+1)_k + a_{22} z(n+1)_k. \]
By substituting it into the multirate formula (6), the slow component \( z \) can be written as
\[ Z(n+1)_k = M_s Z_{nk} + \Theta_s Y(n+1)_k, \]
where \( M_s = (I + H a_{22} e_2 e_1^T) \Phi_s, \quad \Theta_s = H \varepsilon e_2 e_1^T. \)

Linear interpolation. From (5) the fast component \( y \) can be written as
\[ Y_{\sigma+1} = \Phi_f Y_{\sigma} + \Gamma_f (h a_{11} e_1^T Y_{\sigma+1} + h \mu_{eff} e_1^T \tilde{Z}_{\sigma+1}), \]
where \( \sigma = nk + i \) for \( i = 0, 1, \ldots, k - 1. \)
To obtain (14) a little manipulation is needed
\[ Y_{\sigma+1} = M_f Y_{\sigma} + \Theta_f \tilde{Z}_{\sigma+1}, \]
and
\[ M_f = \left( I + \frac{h a_{11}}{1 - \beta_{eff} h a_{11}} \Gamma_f e_1^T \right) \Phi_f, \quad \Theta_f = \frac{h \mu_{eff}}{1 - \beta_{eff} h a_{11}} \Gamma_f e_1^T. \]

To complete the description of \( Y_{\sigma+1} \) when \( i = k - 1 \), we need some preliminary results.
From the sparsity patterns of \( \Theta_f \) and \( \Theta_s \); see (12) and (15), we have \( \Theta_f \Theta_s = 0 \). Therefore from the linear polynomial (9), it follows that
\[ \Theta_f \tilde{Z}_{\sigma+1} = \Theta_f M_i Z_{nk}. \]
By substituting (16) into (14) the fast component may be written as
\[ Y_{\sigma+1} = M_f Y_{\sigma} + \Theta_f M_i Z_{nk}, \]
where \( \sigma = nk + i \) for \( i = 0, 1, \ldots, k - 1. \)
An obvious induction on \( i \), and some tedious manipulation yields
\[ Y(n+1)_k = M_f^k Y_{nk} + S_f Z_{nk}, \]
where
\[ S_f = \sum_{i=0}^{k-1} M_f^{k-(i+1)} \Theta_f M_i, \]
and the matrix \( M_i \) is given by (10).
The complete description of the slow variable is obtained by substituting (18) into (11)
\[ Z(n+1)_k = \Theta_s M_f^k Y_{nk} + (M_s + \Theta_s S_f) Z_{nk}, \]
Thus a compound step of the linear multirate method based on full linear interpolation is given by
\[ \begin{pmatrix} Y(n+1)_k \\ Z(n+1)_k \end{pmatrix} = \begin{pmatrix} M_f^k & S_f \\ \Theta_s M_f^k & M_s + \Theta_s S_f \end{pmatrix} \begin{pmatrix} Y_{nk} \\ Z_{nk} \end{pmatrix}. \]
where $M_s, \Theta_s$ as in (12), $M_f, \Theta_f$ as in (15), and $S_f$ as in (19).

Forward information. Now, we define $\hat{z}$ as

$$\hat{z}_\sigma = z_{(n+1)k}, \quad \text{for all } x.$$ 

Then $\hat{f}_\sigma = f(x_\sigma, y_\sigma, z_{(n+1)k})$, and $\sigma = nk + i$ for $i = 0, 1, \ldots, k - 1$.

By using similar argument as in the case of linear interpolation, it can be shown that a compound step of the linear multirate method using zero–order interpolation (forward information) is given by

$$
\begin{pmatrix}
M_{f}^k & F_f \Theta_f M_s & Y_{nk} \\
\Theta_s M_f^k & (I + \Theta_s F_f \Theta_f) M_s & Z_{nk}
\end{pmatrix},
$$

where $M_s, \Theta_s$ as in (12), $M_f$ and $\Theta_f$ as in (15), $I$ is the identity matrix, and

$$F_f = \sum_{l=0}^{k-1} M_f^l.$$  

(21)

Backward information. In this case we define $\hat{z}$ as

$$\hat{z}_\sigma = z_{nk}, \quad \text{for all } x,$$

and $\hat{f}_\sigma = f(x_\sigma, y_\sigma, z_{nk})$, $\sigma = nk + i$, $i = 0, 1, \ldots, k - 1$.

In a similar way as in above two cases, a compound step of the linear multirate method using zero–order interpolation (backward information) is given by

$$
\begin{pmatrix}
Y_{(n+1)k} \\
Z_{(n+1)k}
\end{pmatrix} = \begin{pmatrix}
M_{f}^k & F_f \Theta_f M_s & Y_{nk} \\
\Theta_s M_f^k & M_s + \Theta_s F_f \Theta_f & Z_{nk}
\end{pmatrix},
$$

where $M_s, \Theta_s$ as in (12), $M_f, \Theta_f$ as in (15), and $F_f$ is given by (21).

Corollary 2.3. The companion matrix $M$ of the semi–implicit multirate linear method defined by (5), (6) can be decomposed into the sum of two matrix: an uncoupled matrix and a coupled matrix

Linear interpolation

$$
\begin{pmatrix}
M_{f}^k & 0 \\
0 & M_s
\end{pmatrix} + \begin{pmatrix}
0 & S_f \\
\Theta_s M_f^k & \Theta_s S_f
\end{pmatrix}.
$$

Forward information

$$
\begin{pmatrix}
M_{f}^k & 0 \\
0 & M_s
\end{pmatrix} + \begin{pmatrix}
0 & F_f \Theta_f M_s \\
\Theta_s M_f^k & \Theta_s F_f \Theta_f M_s
\end{pmatrix}.
$$

Backward information

$$
\begin{pmatrix}
M_{f}^k & 0 \\
0 & M_s
\end{pmatrix} + \begin{pmatrix}
0 & F_f \Theta_f \\
\Theta_s M_f^k & \Theta_s F_f \Theta_f
\end{pmatrix}.
$$

We are now in position to consider the special case of a semi–implicit multirate linear multistep method with $p = 1$. From Theorem 2.2 the order of the companion matrix $M$ must be 4. However if we consider only the components $y_{nk}, z_{nk}$ from $Y_{nk}$ and $Z_{nk}$; then, we can reduce the companion matrix order to 2, and we obtain from Theorem 2.2 the following result.
Corollary 2.4. Suppose that we use a semi–implicit multirate linear method defined by (5), (6) with \( p = 1 \) to solve the linear scalar test equation (4). Assume further that full linear interpolation or zero–order interpolation is used to approximate the values of the slow variable \( z \) that are not in the grid. Then, the companion matrix \( \mathcal{M} \) is of order 2 and has the following general structure for all the former integration methods

\[
\mathcal{M} = \begin{pmatrix} P_f^k + m_{11} \varepsilon & m_{12} \mu \\ m_{21} \varepsilon & P_s \end{pmatrix},
\]

where \( P_f = P_f(ha_{11}) \), \( P_s = P_s(Ha_{22}) \), are the stability functions of the integrating methods used for the fast and slow subsystems respectively. The elements \( m_{ij} \), \( i, j = 1, 2 \) depend on the interpolating polynomial used, and \( m_{21} = \beta_0 H \) in all cases. The other elements are given by

Linear interpolation

\[
m_{11} = \frac{Hh}{1 - \beta_1^* ha_{11}} \left( \frac{\zeta (P_f^k + (1 - P_f)k - 1)}{(1 - P_f)^2 k} + \beta_1^* \right),
\]
\[
m_{12} = \left( \frac{\beta_0 P_f^k}{\alpha_0^* + \beta_0^* ha_{11}} \right) + \frac{1}{1 - \beta_1^* ha_{11}} \left( \frac{\zeta (1 - P_f^{-1})}{1 - P_f} + \beta_1^* \right) + \frac{m_{11}}{Hh} h,
\]

where

\[
\zeta = \frac{\beta_0^* + \alpha_0^* \beta_1^*}{\alpha_0^* + \beta_0^* ha_{11}} P_f.
\]

Forward information

\[
m_{11} = \frac{(\beta_0^* + \beta_1^*) (1 - P_f^k) \beta_0 H h}{(1 - \alpha_0^*) - (\beta_0^* + \beta_1^*) ha_{11}},
\]
\[
m_{12} = \frac{m_{11}}{\beta_0 H} P_s.
\]

Backward information

\[
m_{11} = 0,
\]
\[
m_{12} = \frac{(\beta_0^* + \beta_1^*) (1 - P_f^k) \beta_0}{(1 - \alpha_0^*) - (\beta_0^* + \beta_1^*) ha_{11}}.
\]

When the stability conditions are derived by applying the Routh-Hurwitz criteria to the characteristic polynomial of the companion matrix, a large number of algebraic manipulations are necessary. To reduce them, we derive a simple set of results that are directly related to the companion matrix rather than to the characteristic polynomial.

Lemma 2.5 is a reformulation of the Routh-Hurwitz conditions. It gives us necessary and sufficient conditions to know when the companion matrix \( \mathcal{M} \) of a semi–implicit multirate linear method with \( p = 1 \) satisfies that \( \rho(\mathcal{M}) < 1 \). Consequently, the absolute stability regions of this type of multirate methods can be characterized by a simple criterion.

Lemma 2.5. Let \( A \) be a square real matrix of order two

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

Then \( \rho(A) < 1 \) if and only if the following three conditions are satisfied
(i) \(-(\text{tr}(A) + \text{det}(A)) < 1, \\
(ii) \text{det}(A) < 1, \\
(iii) \text{tr}(A) - \text{det}(A) < 1,

where \text{tr}(A) = a + b is the trace of \(A\) and \text{det}(A) is the determinant of \(A\).

The proof is a consequence of the Routh–Hurwitz criteria [2].

3. Numerical Computation of Absolute Stability Regions

In this section we develop a procedure to plot the absolute stability regions of the multirate methods defined by (5), (6), when \(p = 1\). In order to display these stability regions, we present diagrams showing the boundary \(\partial \mathcal{R}\) of the region, plotted in the \(R^2\) plane. This procedure relies on two fundamental points: Corollary 2.4 that gives us the way to obtain the companion matrix of the multirate method that we considered, and Lemma 2.5 that gives us the necessary and sufficient conditions to know when the spectral radius of this companion matrix is less than one.

Consider the \(2 \times 2\) linear scalar test system given by Equation (4). We approximate its solution by means of the forward Euler multirate (FEM) defined by (5), (6), with micro step \(h\), macro step \(H = kh\) and \(p = 1\). To approximate the values of \(z\) that are not in the mesh, we use the three numerical strategies developed in Section 2. Namely, we base the multirate discretization on linear interpolation (FEMLI), forward information (FEMFI), or backward information (FEMBI).

From Corollary 2.4, we obtain the structure of the companion matrix of these methods. The coefficients \(m_{11}, m_{12}\) are given in Table I.

<table>
<thead>
<tr>
<th>Strategies</th>
<th>(m_{11})</th>
<th>(m_{12})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear interpolation</td>
<td>((P^f_k - Ha_{11} - 1)/a_{11}^2) (+ a_{22}(P^f_k - Ha_{11} - 1)/a_{11}^2)</td>
<td></td>
</tr>
<tr>
<td>Forward information</td>
<td>(H(P^f_k - 1)/a_{11})</td>
<td>(P_s(P^f_k - 1)/a_{11})</td>
</tr>
<tr>
<td>Backward information</td>
<td>0</td>
<td>((P^f_k - 1)/a_{11})</td>
</tr>
</tbody>
</table>

In the three former strategies, the stability functions of the forward Euler multirate method for the fast and slow components are \(P_f = 1 + ha_{11}\), and \(P_s = 1 + Ha_{22}\) respectively. The parameter \(m_{21}\) is equal to \(H\) in all cases.

To plot the absolute stability regions, we have to reduce the number of variables at the companion matrix. Therefore, it is convenient to introduce the following change of variables

\[
\alpha = \frac{a_{22}}{a_{11}}, \quad \beta = \frac{\varepsilon \mu}{a_{22}a_{11}}, \quad P_s = 1 - k(1 - P_f)\alpha. \quad (22)
\]
The parameter $\alpha$ can be viewed as a measure of the system's stiffness. To understand the meaning of $\beta$, note that

$$\beta = 1 - \frac{\lambda_y \lambda_z}{a_{11} a_{22}},$$

where $\lambda_y$, $\lambda_z$ are the eigenvalues of the Jacobian matrix of the test equation (4). Thus, $\beta$ gives us the measure for the coupling between the equations, and the magnitude of the coupling degree can be defined as $C_d = |\beta|100\%$.

From Lemma 2.5 the spectral radius satisfies the condition $\rho(M) < 1$, if the following criteria are satisfied

**Linear interpolation**

\[
(1 - P_s)((1 - P^h_f)(1 + \alpha) - 2)\beta + 1 + P^h_f \\
+ (1 + P_s)(1 - P^h_f)\alpha \beta - (1 + 2P^h_f) < 1,
\]

\[
P^h_f P_s - (1 + (P_s(1 - P^h_f)\alpha + (1 - P_s)((1 - P^h_f)(1 + \alpha) - 1))\beta < 1, \\
1 - (1 - P^h_f)(1 - \beta)k\alpha < 1.
\]

**Forward information**

\[
((1 + P^h_f) - (1 - P^h_f)\beta)(1 - P_f)k\alpha - (1 + 2P^h_f) < 1, \\
P^h_f P_s < 1,
\]

\[
1 - (1 - P^h_f)(1 - \beta)k\alpha < 1.
\]

**Backward information**

\[
(1 - P_f)(1 + \beta + (1 - \beta)P^h_f)k\alpha - (1 + 2P^h_f) < 1, \\
P^h_f - (1 - P_f)((1 - P^h_f)\beta + P^h_f)k\alpha < 1, \\
1 - (1 - P^h_f)(1 - P_f)(1 - \beta)k\alpha < 1.
\]

Consequently, the absolute stability regions belong to the space defined by the parameters $(P_f, \beta, \alpha, k)$: a four dimension space. To deal with this problem we use the canonical projection of $\mathbb{R}^4$ onto $\mathbb{R}^2$.

We must select the micro step $h > 0$ for the fast component, in such a way that the stability polynomial satisfies $|P_f(h a_{11})| < 1$. As a consequence, we have that $(1 - P^h_f) > 0$ and $(1 - P_f) > 0$. Furthermore, since $k$ is a positive integer, from (25), (28) and (31) we need to analyze two cases:

$$\beta = \begin{cases} 
\beta < 1 & \text{if } \alpha > 0, \\
\beta > 1 & \text{if } \alpha < 0.
\end{cases}$$

To display the absolute stability regions, we perform the already mentioned projection from $\mathbb{R}^2$ to $\mathbb{R}^4$. We hold fixed the parameter $\alpha$ and the multirate factor $k$. Then, we determine the location of the boundary, by varying the parameters $\beta$ and $P_f$, verifying whether the conditions (23) to (31) are satisfied or not, in each case. Using these results, we plot the boundary $\partial R$ region of absolute stability in the plane $P_f \beta$.

Figures 1 to 3 display the absolute stability regions comparison of the three multirates methods FEMLI, FEMFI, FEMBI. These methods are stable to the right of their boundaries,
and unstable to left of them. The $y$–axis represents the coupling factor $\beta$ and the $x$–axis represents the fast stability function $P_f$. The multirate factor $k$ and the stiffness factor $\alpha$ are constants for each diagram in these figures.

To show stability regions with different coupling degrees varying from weak coupling to strongly coupled, we choose three stiffness factors: $\alpha = 6.29 \times 10^{-4}$, $\alpha = 2$ and $\alpha = -0.1$. We can see strongly coupled regions when $\alpha = 6.29 \times 10^{-4}$. The second, $\alpha = 2.0$, permit us observe weak coupling regions. Finally, when $\alpha = -0.1$, we can display absolute stability regions for $\beta > 1$, and they are medium strong coupling regions.

The multirate factors, $k = 2$ and $k = 8$, are selected to show how the stability regions tend to contract when $k$ is increased.

Figures 1 and 2 show the absolute stability regions for the stiffness factor $\alpha > 0$ ($\beta < 1$), and Figure 3 shows the absolute stability regions for $\alpha < 0$ ($\beta > 1$).

These regions become smaller when $|\beta|$ or $k$ are increased. The absolute stability regions tend to shrink toward the right vertical line of the diagrams of Figures 1 to 3 when $|\beta| \rightarrow \infty$ or $k \rightarrow \infty$. In some cases, in the stronger coupling stability region, the forward Euler multirate method based on zero–order interpolation has a larger stability region than the forward Euler multirate based on full linear interpolation; see 1(a), 1(b) ($\beta = -3 \times 10^{3}$), and 3(a), 3(b) ($\beta = 38$).

Besides, it was possible to corroborate the theoretical prediction of the existence, for the forward Euler multirate method, of absolute stability regions for stiffness factor $\alpha > 0$ and $\alpha < 0$; see $\beta$ cases (32) and Figures 1, 2, when $\alpha > 0$ and Figure 3 when $\alpha < 0$.

It may seems natural to expect that if a system of ordinary differential equations can be integrated with a multirate method by using zero–order interpolation, it can be integrated with the same multirate method but using linear interpolation. However, the following result contradicts the previous position.

When the stiffness factor is positive and small, $\alpha \approx 10^{-4}$, the stability domain includes very large coupling, $|\beta| \approx 10^{3}$; see Figure 1. In the strongest coupling region, $\beta = -3 \times 10^{3}$, the method that has a larger stability region is FEMFI, the next is FEMLI and finally FEMBI. The foregoing is fulfilled for $k = 2$ and $k = 8$. Thus, FEMFI could integrate strongly coupling systems of ordinary differential equations that FEMLI could not integrate them.

Moreover, from Figure 1 it can be concluded that the forward Euler multirate method based on full linear interpolation or zero–order interpolation is stable for large magnitudes of $\beta$. This consequently opens the possibility of integrating strongly coupled systems of differential equations by using semi–implicit multirate linear multistep methods.

When $\alpha = 2$; see Figure 2, the coupling lies between $-2 < \beta < 1$. Outside of this range the stability function $P_f$ is sufficiently close to 1 and the stable schemes for FEMLI, FEMFI and FEMBI are obtained with small micro steps $h$. In the strongest coupling region, $\beta = -2$, FEMLI has the largest stability region.

If $\alpha < 0$ the coupling factor $\beta$ is positive and it lies between $1 < \beta < 38$; see Figure 3. Outside of this range, if we want to have stable schemes for the three multirate methods FEMLI, FEMFI and FEMBI we need small micro steps $h$. In the strongest coupling region, $\beta = 38$, FEMFI has the largest region stability.
4. Conclusions

In this paper we analyzed the absolute stability conditions for the class of semi-implicit multirate linear multistep methods based on full linear polynomial interpolation or zero-order interpolation. We used a two-dimensional scalar test equation to perform the stability analysis.

The companion matrices of the multirate methods are the starting point of the stability
analysis. However, the characterization of the stability regions involves a considerable number of algebraic manipulations. To simplify the stability analysis, we found a common algebraic structure for these matrices. Besides, we obtained a simple set of stability conditions, which are directly related to the companion matrix rather than to the characteristic polynomial.

A main contribution of this approach is that we obtained the general structure of the companion matrices corresponding to the semi–implicit multirate linear multistep methods based on full linear interpolation or zero–order interpolation. In addition, we showed that these three companion matrices can be reduce to a single two–dimensional scalar matrix, when \( p = 1 \).

Using the above results, we provided an explicit characterization of the stability conditions. In the one–step discretization case, we obtained a necessary and sufficient computationally simple criterion for determining when a linear system is in the absolute stability region of a given multirate linear semi–implicit method. In addition, we showed that the former companion matrices are decomposed in an uncoupled matrix plus a coupled matrix.

We developed a procedure to plot the absolute stability regions, for the semi–implicit multirate linear multistep methods, when \( p = 1 \). This procedure relies on two fundamental points: a) an easy way of obtaining the companion matrix of the method that we are considering, and b) to know the necessary and sufficient conditions that the companion matrix must satisfy so that its spectral radius is less than one. Corollary 2.4 and Lemma 2.5 give us these conditions.

We plotted in the plane \( P_f/\beta \) the boundary of the absolute stability region, for the forward Euler multirate method based on full linear interpolation or zero–order interpolation, for different values of \( \alpha \) and \( k \). These regions, become smaller when, \( |\beta| \) or \( k \) are increased. In some cases, the forward Euler multirate method based on zero–order interpolation, had a larger stability region than the forward Euler multirate based on full linear interpolation. This is a surprising counter intuitive result.

Figure 3. Absolute stability regions for the FEM with \( \alpha = −0.1 \)
Additionally, we corroborated the theoretical prediction of the existence, for the forward Euler multirate method, of absolute stability regions for stiffness factor $\alpha > 0$ and $\alpha < 0$, and we concluded that the forward Euler multirate method based on full linear interpolation or zero–order interpolation is stable for large magnitudes of $\beta$. This consequently opens the possibility of integrating strongly coupled systems of differential equations by using semi–implicit multirate linear multistep methods.

It is interesting to mention that the procedure developed to plot the absolute stability regions of the multirate linear schemes, when $p = 1$, is directly related to the companion matrix rather than to the characteristic polynomial. The absolute stability regions that we obtained by verifying the correctness of the method.

A possible generalization of the results presented in this work, can be stated en the following conjecture:

Conjecture A. Suppose that we use a multirate linear multistep method defined by (2) and (3) with $p = 1$ to solve the linear scalar test equation (4). Assume further that full linear interpolation or zero–order interpolation is used to approximate the values of the slow variable $z$ that are not in the grid. Then, the companion matrix $M$ is of order 2 and has the following general structure for all the former integration methods

$$M = \begin{pmatrix} P_f & m_{12}\mu \\ m_{21}\varepsilon & P_s + m_{22}\varepsilon\mu \end{pmatrix},$$

where $P_f = P_f(ha_{11})$, $P_s = P_s(Ha_{22})$, are the stability functions of the integrating methods used for the fast and slow subsystems respectively. The elements $m_{ij}$, $i, j = 1, 2$ depend on the interpolating polynomial used.

Clearly further research is needed. An obvious generalization is to include in the stability analysis semi–implicit multirate linear two–step methods. In this case, the algebraic manipulations are increased in complexity, since the dimension of the companion matrix is $8 \times 8$. It is also necessary to find new conditions, which are computationally simple for the absolute stability in this case. Furthermore, to reduce the algebraic manipulations these conditions should be related to the companion matrix of the multirate method rather than to the characteristic polynomial.

APPENDIX

Consider the initial value problem (IVP) for the differential equation

$$y'(x) = f(x, y), \quad y(a) = y_0, \quad (33)$$

with the solution required at certain values of $x \in [a, b]$. The prime ($'$) means $d/dx$, and $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Suppose that (33) is solved by the linear multistep method of $p$ steps.

$$y_{n+1} = \sum_{j=0}^{p-1}(\alpha_j y_{n-j} + h\beta_j f_{n-j}) + h\beta_p f_{n+1}, \quad (34)$$

where $\alpha_j, \beta_j, h > 0$ are real constants, and $f_n = f(x_n, y_n)$. We assume that $|\alpha_0| + |\beta_0| \neq 0$, and (34) is consistent and zero–stable [1].
To get flexibility and facilitate the analysis of the multirate algorithms, we introduce some notation as given by Wells in [14]. Define the \((2p + 1) \times 1\) matrix \(a\) by
\[
a = (\alpha_0, \alpha_1, \ldots, \alpha_{p-1}, \beta_0, \beta_1, \ldots, \beta_p)^T,
\]
and let \(\Phi(a)\) be the \(2p \times 2p\) matrix defined by
\[
\Phi(a) = (\tilde{a}^T, 0, e_1^T, \ldots, e_{2p-2}^T)^T
\]
where \(\tilde{a}^T = (\alpha_0, \beta_0, \ldots, \alpha_{p-1}, \beta_{p-1})\) and \(e_k\) is, as usual, the column vector with 1 in position \(k\) and 0 elsewhere.

Let \(\Gamma(a)\) be the \(2p \times 1\) matrix, given by
\[
\Gamma(a) = \beta_p e_1 + e_2,
\]
Thus, by defining \(Y_{n+1}\) as
\[
Y_{n+1} = (y_{n+1}, hf_{n+1}, y_n, hf_n, \ldots, y_{n-p+2}, hf_{n-p+2})^T,
\]
we can write (34) in the form
\[
Y_{n+1} = \Phi Y_n + h\Gamma f_{n+1},
\]
where \(\Phi = \Phi(a)\), \(\Gamma = \Gamma(a)\) are given by (36) and (37).

If \(y' = \lambda y\) is solved by (39) we obtain
\[
Y_{n+1} = \Phi Y_n + h\lambda (\beta_p e_1 + e_2)e_1^T Y_{n+1}.
\]
This implies that
\[
Y_{n+1} = MY_n,
\]
where \(M = (I + (h\lambda/(1 - \beta_p))(\beta_p e_1 + e_2)e_1^T)\Phi\), the matrix \(I\) is the identity matrix. The matrix \(M\) is called the companion matrix.

REFERENCES