Output feedback hybrid force/motion control for robotic manipulators interacting with unknown rigid surfaces

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Abstract

The problem of hybrid force and motion control over unknown rigid surfaces when only joint position measurements are available is considered. To overcome this problem, an extended–state high–gain observer is designed to simultaneously estimate the contact force and joint velocities. These estimated signals are in turn employed to design a local estimator of the unknown surface gradient. This gradient is utilized to decompose the task space into two orthogonal subspaces, one for force tracking and the other one for motion control. A simple position *PID* and force *PI* controllers are proposed to track the desired signals. Finally, a mathematical analysis of the closed loop dynamics is carried out, guaranteeing uniform ultimate boundedness of the position and force tracking errors, and of the surface gradient estimation error. A numerical simulation is employed to validate the approach in an ideal scenario, while experiments are carried out to test the proposed strategy when uncertainties and unmodeled dynamics are present.

1 Introduction

Many applications involving a robotic manipulator require its interaction with the environment. In such a case, it becomes necessary to control not only the motion of the manipulator but also the interaction force with the environment. There are basically two approaches to deal with the motion and force control problem: the direct and the indirect force control. To the later category belong the so called impedance and compliance controllers¹ that achieve the desired interaction force by controlling the motion of the manipulator without an explicit force control loop. This kind of controllers are best suited for compliant environments (see for example Refs.^{2–4}). On the other hand, direct force control comprehends the hybrid force/motion⁵ and the parallel control.^{6,7} The hybrid force/motion control is more suitable when the motion of the manipulator is restricted to a rigid surface (*i.e.*, it can be represented by holonomic constraints),

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since it relies on the decomposition of the task space into motion–controlled and force–controlled subspaces.

For the mentioned approaches it is commonly assumed that kinematic and dynamic models of both manipulator and environment are available. Moreover, it is assumed that positions, velocities and contact force are measured. Several solutions have been proposed to deal with uncertainties on the surface geometry, including visual identification,⁸⁻¹¹ force-based reconstruction,¹² robustness against the kinematic and/or dynamic uncertainties,¹³⁻¹⁵ and adaptive control.¹⁶⁻²¹ In the case of compliant environments the *a priori* exact description of the environment constraints is not always necessary, since the controllers can be designed to be robust against uncertainties in this description. In contrast, it has been shown that the geometric description of the constraints plays a critical role for the rigid surfaces case, since uncertainties in this description can easily lead to instability of the closed loop.²²

The basic idea behind the mentioned schemes when no exact description of the geometry of the constraints is available, is to reconstruct the surface by means of position, velocity, visual, and/or force measurements. Nevertheless, in many applications it is desired to have the least number of sensors for a variety of reasons (reduction of weight, size, costs, etc.). Therefore, another interesting research direction is the output feedback motion and force control, *i.e.*, when only position measurements are available. Some solutions have been proposed to solve this problem for compliant surfaces, *e.g.*, an open loop force control with adaptive identification of the surface,²³ extended active observers,²⁴ and sliding mode observers,²⁵ among others. For the rigid contact case, some solutions based on linear observers,^{26,27} nonlinear PID observer–based control,²⁸ and Generalized Proportional Integral observers²⁹ have been proposed as well.

On the other hand, the mentioned output feedback controllers over a rigid surface also need a precise description of the holonomic constrains, which is commonly obtained by means of kinematic calibration. However, there are some scenarios where this description cannot be obtained *a priori*, which represents an important disadvantage. Application examples of force control over an unknown environments that need the identification of the surface include exploration of unknown and dark spaces (tank, pipes, planets).¹⁹ Regarding this problem, in ref. [30] a fault detection and isolation technique is employed to detect uncalibrated obstacles and then a reconstruction of the contact force is utilized to regulate this force exerted over the unknown environment. The mentioned algorithm does not require force measurement nor acceleration but only joint positions and velocities.

In this work, an output feedback hybrid force/motion tracking controller for robotic manipulators subject to holonomic constraints with a local online estimation of the contact surface is proposed. The work is based on the GPI observers methodology introduced in ref. [31] and further developed in ref. [29] to the case of contact force estimation. To the best of the authors' knowledge there are no similar results for the problem treated here, *i.e.*, the hybrid force/motion control without force/velocity and/or vision measurements and without a geometric description of the surface. Although the main idea is similar to ref. [30] there are some differences and advantages in this work. The first difference is that, in contrast with ref. [30], no joint velocities are needed to implement the controller. A second distinction of our work is that the proposed surface gradient estimator is designed to be less sensitive to noise due its filtering nature, which is an important practical advantage. Finally, in this work a mathematical analysis is carried out, which gives conditions on the controller and observer parameters to guarantee the achieving of the control and estimation objective and the boundedness of all signals of interest.

The main contribution of this work is the design and implementation of a control algorithm that simultaneously estimates the joint velocities, the contact force, and the contact surface using only joint position measurements, while following a desired trajectory both in position and in contact force. To the best of the authors' knowledge this work is the first attempt to solve the problem presented here, *i.e.* the closed–loop hybrid force/motion control over unknown surfaces using only joint position measurements.

The proposed approach guarantees ultimate boundedness of all the signals of interest, with arbitrarily small ultimate bounds, which means that force and position tracking, velocity and force reconstruction and surface identification can be achieved with arbitrary precision in the ideal case. This is supported by the numerical simulations presented below. In a real scenario, this ultimate bound is no longer arbitrarily small due to several reasons, in particular to contact friction, which is assumed to be negligible for the control design and the stability analysis. Nevertheless, the experimental results presented below are in accordance with the expected performance.

This paper is organized as follows: in Section 2 a mathematical model of the system and some useful properties are presented. In Section 3 the main contribution of this work is developed, while in Section 4 both numerical simulations and experiments are shown to illustrate the effectiveness of the approach. Some concluding remarks and directions for future work are given in Section 5.

2 Mathematical model and properties

Consider a *n*-degrees of freedom manipulator in contact with a rigid surface. Let $q \in \mathbb{R}^n$ be the vector of generalized coordinates and $\tau \in \mathbb{R}^n$ the vector of input torques. The corresponding dynamic model is given by³²

$$H(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + g(q) = \tau + J_{\omega}^{T}(q)\lambda, \qquad (1)$$

where $H(q) \in \mathbb{R}^{n \times n}$ is the inertia matrix, $C(q, \dot{q})\dot{q} \in \mathbb{R}^n$ is the vector of Coriolis and centripetal torques, $D \in \mathbb{R}^{n \times n}$ is the diagonal matrix of viscous friction coefficients, $g(q) \in \mathbb{R}^n$ is the vector of gravity torques, $\lambda \in \mathbb{R}^m$ is a vector of Lagrange multipliers (physically represents the force exerted by the manipulator over the environment at the contact point), and $J_{\varphi}(q) \triangleq \partial \varphi(q) / \partial q \in \mathbb{R}^{m \times n}$ is the gradient of the *m* holonomic constraints, specified in terms of the generalized coordinates, defined by

$$\varphi(q) = \mathbf{0}. \tag{2}$$

These constraints can also be defined in terms of the end effector coordinates $\boldsymbol{x} \in \mathbb{R}^n, i.e.$

$$\varphi(\boldsymbol{x}) = \boldsymbol{0} \,. \tag{3}$$

The gradient of this constraint $J_{\varphi \mathbf{x}} \triangleq \partial \varphi(\mathbf{x}) / \partial \mathbf{x} \in \mathbb{R}^{m \times n}$ is related to $J_{\varphi}(\mathbf{q})$ by

$$\boldsymbol{J}_{\varphi}(\boldsymbol{q}) = \boldsymbol{J}_{\varphi \mathbf{x}} \boldsymbol{J}(\boldsymbol{q}) \,, \tag{4}$$

where $J(q) \in \mathbb{R}^{n \times n}$ is the analytic Jacobian of the manipulator. Note that with a suitable normalization it can always be obtained $\|J_{\varphi x}\| = 1$. In this paper it is assumed that this is the case. In addition, it is made the following rather standard assumption, which should be taken into account at the trajectory planning stage.

Assumption 2.1 The manipulator never reaches a singularity, so $J^{-1}(q)$ always exists ¹.

Now there are listed some useful properties of the model (1). Notice that for simplicity's sake it is assumed that the robot has only revolute joints.

Property 2.1 The inertia matrix H(q) is symmetric positive definite and $\forall q, y \in \mathbb{R}^n$ it holds $\lambda_h \|y\|^2 \leq y^T H(q) y \leq \lambda_H \|y\|^2$, with $0 < \lambda_h \leq \lambda_H < \infty$.

Property 2.2 With a proper definition of $C(q, \dot{q})$, the matrix $\dot{H}(q) - 2C(q, \dot{q})$ is skew symmetric.

Property 2.3 The vector C(q, x)y satisfies $C(q, x)y = C(q, y)x, \forall x, y \in \mathbb{R}^n$.

When the tip of the robot is in contact with a rigid surface, a local decomposition of the task space can be done as follows. Let $J_{\varphi}^{+} = J_{\varphi}^{\mathrm{T}} \left(J_{\varphi} J_{\varphi}^{\mathrm{T}} \right)^{-1} \in \mathbb{R}^{n \times m}$, $P(q) = J_{\varphi}^{+} J_{\varphi} \in \mathbb{R}^{n \times n}$, and $Q(q) = I_{n \times n} - P(q) \in \mathbb{R}^{n \times n}$, whence Q and Pare orthogonal projection matrices, *i.e.*, QP = O, $QJ_{\varphi} = O$, and $J_{\varphi}^{\mathrm{T}}Q = O$. Furthermore, QQ = Q and PP = P. Then, it can be stated the following.

Property 2.4 The vector \dot{q} can be written as

$$\dot{\boldsymbol{q}} = \boldsymbol{Q}(\boldsymbol{q})\dot{\boldsymbol{q}} + \boldsymbol{P}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{Q}(\boldsymbol{q})\dot{\boldsymbol{q}}.$$
(5)

Finally, an useful fact concerning the position tracking error vector and the subspaces projected by Q and P is stated. Let $q_d \in \mathbb{R}^n$ be the desired position in articular coordinates, then the tracking error $e \triangleq q - q_d$ satisfies the following.³⁴

 $^{^{1}}$ This condition should be checked at the robot initial configuration and at the trajectory generation stage, what can be done by computing the manipulability ellipsoids as shown in ref. [33, Sec. 3.9]

Fact 2.1 Assume $q_{\rm d}$ is designed to satisfy $\varphi(q_{\rm d}) = 0$. Whenever the manipulator is restricted to fulfil the constraint (2) and the tracking error e is sufficiently small, the following approximation can be made

$$e \approx Q(q)e$$
, (6)

because the error tends to be contained in the tangent subspace at the point q projected by Q(q). Furthermore, from Property 2.4 it follows

$$\dot{\boldsymbol{q}}_{\mathrm{d}} \approx \boldsymbol{Q}(\boldsymbol{q}) \dot{\boldsymbol{q}}_{\mathrm{d}} \implies \dot{\boldsymbol{e}} \approx \boldsymbol{Q}(\boldsymbol{q}) (\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_{\mathrm{d}}) = \boldsymbol{Q}(\boldsymbol{q}) \dot{\boldsymbol{e}},$$

$$(7)$$

Remark 2.1 As illustrated in Figure 1 of ref. [34], which we reproduce here as Figure 1. Fact 2.1 does not necessarily imply $e \approx 0$. Roughly speaking, it says that the error is contained mostly in the subspace projected by Q(q) in a neighbourhood of e = 0. Furthermore, for a planar surface it is trivially satisfied.



Figure 1: Illustration of Fact 2.1: a) error e not contained in the subspace projected by Q, b) error e contained in the subspace projected by Q.

3 Main result

Consider the case when the constraint is one-dimensional (*i.e.*, m = 1), so that $\lambda = \lambda \in \mathbb{R}$ represents the scalar force that exerts the manipulator over the surface and $J_{\varphi_{\mathbf{X}}}$ maps every vector in the Cartesian space, into the normal to the tangent plane to the surface at the contact point.

Let $q_1 \triangleq q$ and $q_2 \triangleq \dot{q}$. A state space representation of model (1) is given by

$$\dot{\boldsymbol{q}}_1 = \boldsymbol{q}_2 \tag{8}$$

$$\dot{\boldsymbol{q}}_2 = \boldsymbol{H}^{-1}(\boldsymbol{q}_1) \left(\boldsymbol{\tau} - \boldsymbol{N}(\boldsymbol{q}_1, \boldsymbol{q}_2) \right) + \boldsymbol{z}_1 \,,$$
(9)

where $N(q_1, q_2) \triangleq C(q_1, q_2)q_2 + Dq_2 + g(q_1)$ and

$$\boldsymbol{z}_1 \triangleq \boldsymbol{H}^{-1}(\boldsymbol{q}_1) \boldsymbol{J}_{\varphi}^{\mathrm{T}}(\boldsymbol{q}_1) \boldsymbol{\lambda} \,. \tag{10}$$

One of the goals of this work is to estimate the contact force λ , contained in the variable z_1 . We make the following assumptions concerning z_1 .³¹

Assumption 3.1 The vector z_1 can be locally modelled as a vector of Taylor polynomials in the time variable t, plus a residual term i.e.,

$$\boldsymbol{z}_{1}(t) = \sum_{i=1}^{p} \boldsymbol{a}_{i} t^{i} + \boldsymbol{r}(t) , \qquad (11)$$

where $a_i \in \mathbb{R}^n, i = 1, ..., p$ are vectors of constant coefficients and $r(t) \in \mathbb{R}^n$ is the residual term.

Assumption 3.2 At least the first p time derivatives of z_1 exist.

Assumptions 3.1 and 3.2 allow us to write an internal model of \boldsymbol{z}_1 for the foregoing estimation as

$$\dot{\boldsymbol{z}}_1 = \boldsymbol{z}_2 \tag{12}$$

$$\dot{\boldsymbol{z}}_2 = \boldsymbol{z}_3 \tag{13}$$

$$\dot{\boldsymbol{z}}_{p-1} = \boldsymbol{z}_p \tag{14}$$

$$\dot{\boldsymbol{z}}_p = \boldsymbol{r}^{(p)}(t) \,. \tag{15}$$

Taking into account this model, it is proposed the extended state observer³¹

:

$$\dot{\hat{\boldsymbol{q}}}_1 = \hat{\boldsymbol{q}}_2 + \boldsymbol{\lambda}_{p+1} \tilde{\boldsymbol{q}}_1 \tag{16}$$

$$\dot{\hat{\boldsymbol{q}}}_{2} = \boldsymbol{H}^{-1}(\boldsymbol{q}_{1})\left(\boldsymbol{\tau} - \boldsymbol{N}(\boldsymbol{q}_{1}, \hat{\boldsymbol{q}}_{2})\right) + \hat{\boldsymbol{z}}_{1} + \boldsymbol{\lambda}_{p}\tilde{\boldsymbol{q}}_{1}$$
(17)

$$\dot{\hat{z}}_1 = \hat{z}_2 + \lambda_{p-1} \tilde{q}_1 \tag{18}$$

$$\dot{\hat{\boldsymbol{z}}}_2 = \hat{\boldsymbol{z}}_3 + \boldsymbol{\lambda}_{p-2} \tilde{\boldsymbol{q}}_1 \tag{19}$$

$$\dot{\hat{\boldsymbol{z}}}_{p-1} = \hat{\boldsymbol{z}}_p + \boldsymbol{\lambda}_1 \tilde{\boldsymbol{q}}_1 \tag{20}$$

$$\dot{\tilde{\boldsymbol{z}}}_p = \boldsymbol{\lambda}_0 \tilde{\boldsymbol{q}}_1 \,, \tag{21}$$

where $\tilde{q}_1 \triangleq q_1 - \hat{q}_1$ and $N(q_1, \hat{q}_2) \triangleq C(q_1, \hat{q}_2) \hat{q}_2 + D\hat{q}_2 + g(q_1)$. Note that \hat{q}_2 is employed instead of q_2 to avoid velocity measurements.

From (4) and (10) it follows

÷

$$\boldsymbol{J}_{\varphi \mathbf{x}}^{\mathrm{T}} \boldsymbol{\lambda} = \boldsymbol{J}^{-\mathrm{T}}(\boldsymbol{q}_{1}) \boldsymbol{H}(\boldsymbol{q}_{1}) \boldsymbol{z}_{1} \,.$$
(22)

As mentioned above, the Lagrange multiplier λ represents the contact force and $\|\boldsymbol{J}_{\varphi \mathbf{x}}^{\mathrm{T}}\|$ is a unitary vector in the normal direction to the surface. For unidirectional constraints it is $\lambda \geq 0$. By taking the Euclidean norm in both sides of (22) an estimate of the contact force can be obtained by

$$\hat{\boldsymbol{\lambda}} = \|\boldsymbol{J}_{\varphi \mathbf{x}}^{\mathrm{T}} \hat{\boldsymbol{\lambda}}\| = \|\boldsymbol{J}^{-\mathrm{T}}(\boldsymbol{q}_{1}) \boldsymbol{H}(\boldsymbol{q}_{1}) \hat{\boldsymbol{z}}_{1}\|.$$
(23)

Because it is assumed that the geometry of the constraint surface is unknown, an online estimation of the gradient of this surface in workspace coordinates is proposed as

$$\dot{\hat{\boldsymbol{J}}}_{\varphi \mathbf{x}}^{\mathrm{T}} = \left(\frac{\gamma}{\hat{\lambda} + \epsilon}\right) \hat{\boldsymbol{Q}}_{\mathbf{x}} \boldsymbol{J}^{-\mathrm{T}}(\boldsymbol{q}_{1}) \boldsymbol{H}(\boldsymbol{q}_{1}) \hat{\boldsymbol{z}}_{1}, \qquad (24)$$

where $\gamma > 0$ is a scalar adaptation gain, $\epsilon \ll \lambda$ is a (small) positive constant to avoid division by zero and $\hat{\boldsymbol{Q}}_{x} \triangleq \boldsymbol{I} - \hat{\boldsymbol{J}}_{\varphi x}^{\mathrm{T}} \left(\hat{\boldsymbol{J}}_{\varphi x} \hat{\boldsymbol{J}}_{\varphi x}^{\mathrm{T}} \right)^{-1} \hat{\boldsymbol{J}}_{\varphi x}$. Since from the definition of $\hat{\boldsymbol{Q}}_{\varphi x}$, $\hat{\boldsymbol{J}}_{\varphi x} \hat{\boldsymbol{Q}}_{x} = \boldsymbol{0}$, after (24) it is

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\|\hat{\boldsymbol{J}}_{\varphi \mathbf{x}}\|^2 \right) = \frac{\mathrm{d}}{\mathrm{d}t} \left(\hat{\boldsymbol{J}}_{\varphi \mathbf{x}} \hat{\boldsymbol{J}}_{\varphi \mathbf{x}}^{\mathrm{T}} \right) = 2 \hat{\boldsymbol{J}}_{\varphi \mathbf{x}} \dot{\hat{\boldsymbol{J}}}_{\varphi \mathbf{x}}^{\mathrm{T}} = \left(\frac{2\gamma}{\hat{\lambda} + \epsilon} \right) \hat{\boldsymbol{J}}_{\varphi \mathbf{x}} \hat{\boldsymbol{Q}}_{\mathbf{x}} \boldsymbol{J}^{-\mathrm{T}}(\boldsymbol{q}_1) \boldsymbol{H}(\boldsymbol{q}_1) \hat{\boldsymbol{z}}_1 = \boldsymbol{0}$$
(25)

Then, $\|\hat{J}_{\varphi x}\| \equiv 1$ as long as $\|\hat{J}_{\varphi x}(t_0)\| = 1$, which is assumed in this work. Hence, the computation of \hat{Q}_x is simplified to

$$\hat{\boldsymbol{Q}}_{\mathrm{x}} = \boldsymbol{I} - \hat{\boldsymbol{J}}_{\varphi \mathrm{x}}^{\mathrm{T}} \hat{\boldsymbol{J}}_{\varphi \mathrm{x}} \,. \tag{26}$$

Additionally, it is defined $\hat{\boldsymbol{P}}_{\mathrm{x}} \triangleq \boldsymbol{I} - \hat{\boldsymbol{Q}}_{\mathrm{x}}$.

Remark 3.1 Without the term $\hat{Q}_{\mathbf{x}}$, the gradient estimator is just a filtered version of the estimated force vector divided by its module. This idea has been employed in the literature when force measurement is available.^{16, 20, 21} The introduction of $\hat{Q}_{\mathbf{x}}$ is proposed to ensure the norm invariance of $\hat{J}_{\varphi \mathbf{x}}$ as shown in (25).

When dealing with position control over uncertain surfaces, one major issue is trajectory planning. Although the geometry of the environment is considered to be unknown, we make the following assumption to simplify the problem.

Assumption 3.3 The unknown contact surface is smooth, i.e. it can be described by a continuous function (3) with continuous derivatives of all orders with respect to the Cartesian coordinates \mathbf{x} .

We also assume that the tip of the robot is in contact with this surface at $t = t_0$.

Remark 3.2 Although there is no restriction on the concavity, convexity or maximum/minimum gradient of the surface, in a practical application it can be limited by the geometry of the manipulator tip. To deal with the problem of the tip being in contact with the unknown surface at $t = t_0$ without force sensor, the very same force estimator (23) can be employed in free movement to detect a collision, after some predefined threshold (as in ref.[30]).

This way, a locally linear approximation is developed to solve the trajectory planning problem (see ref. [21] for details). Let $\eta > 0$ and let $\boldsymbol{x}_a \in \mathbb{R}^n$ be the output of the unit-gain first order filter

$$\dot{\boldsymbol{x}}_{\mathrm{a}} = -\eta \boldsymbol{x}_{\mathrm{a}} + \eta \boldsymbol{x} \,, \tag{27}$$

with $\boldsymbol{x}_{a}(t_{0})$ the robot tip position at the beginning of the task. For low frequencies, this is equivalent to computing a delayed version of \boldsymbol{x} , *i.e.* $\boldsymbol{x}_{a} \approx \boldsymbol{x}(t-T)$ for some T > 0.

The desired position in Cartesian coordinates $x_d(t) \in \mathbb{R}^n$ can then be computed on line to fulfil

$$\hat{\boldsymbol{\varphi}}(\boldsymbol{x}) = \hat{\boldsymbol{J}}_{\varphi \mathbf{x}}(\boldsymbol{x}_{\mathrm{d}}(t) - \boldsymbol{x}_{\mathrm{a}}) = \boldsymbol{0}.$$
(28)

The corresponding desired position in joint coordinates $\boldsymbol{q}_{\mathrm{d}} \in \mathbb{R}^{n}$ is obtained by solving the inverse kinematics of the manipulator. Finally, let $\lambda_{\mathrm{d}} = \lambda_{\mathrm{d}}(t) \in \mathbb{R}$ be the desired contact force, then it is proposed the observer-based hybrid position/force control

$$\boldsymbol{\tau} = -\boldsymbol{K}_{\mathrm{p}}\boldsymbol{e}_{1} - \boldsymbol{K}_{\mathrm{v}}(\hat{\boldsymbol{q}}_{2} - \dot{\boldsymbol{q}}_{\mathrm{d}}) - \hat{\boldsymbol{Q}}\boldsymbol{K}_{\mathrm{i}} \int_{0}^{t} \boldsymbol{e}_{1} \,\mathrm{d}\vartheta - \hat{\boldsymbol{J}}_{\varphi}^{\mathrm{T}}\lambda_{\mathrm{d}} + \hat{\boldsymbol{J}}_{\varphi}^{+}k_{\mathrm{Fi}}\Delta\bar{F}\,,\qquad(29)$$

where $\mathbf{K}_{\mathrm{p}}, \mathbf{K}_{\mathrm{v}}, \mathbf{K}_{\mathrm{i}} \in \mathbb{R}^{n \times n}$ are diagonal positive definite matrices of constant gains, $k_{\mathrm{Fi}} > 0$ is the integral force control gain, $\mathbf{e}_{1} \triangleq \mathbf{q}_{1} - \mathbf{q}_{\mathrm{d}}$ is the position tracking error, and

$$\hat{\boldsymbol{J}}_{\varphi}^{\mathrm{T}} \triangleq \boldsymbol{J}(\boldsymbol{q})\hat{\boldsymbol{J}}_{\varphi\mathrm{x}}^{\mathrm{T}}$$
(30)

$$\hat{\boldsymbol{J}}_{\varphi}^{+} \triangleq \hat{\boldsymbol{J}}_{\varphi}^{\mathrm{T}} \left(\hat{\boldsymbol{J}}_{\varphi} \hat{\boldsymbol{J}}_{\varphi}^{\mathrm{T}} \right)^{-1}$$
(31)

$$\hat{\boldsymbol{Q}} \triangleq \boldsymbol{I} - \hat{\boldsymbol{J}}_{\varphi}^{+} \hat{\boldsymbol{J}}_{\varphi} \,. \tag{32}$$

Additionally, it is defined

$$\Delta \bar{\lambda} \triangleq \hat{\lambda} - \lambda_{\rm d} \tag{33}$$

$$\Delta \bar{F} \triangleq \int_0^t \Delta \bar{\lambda} \, \mathrm{d}\vartheta \,. \tag{34}$$

Remark 3.3 The proposed observer (16)-(21) requires the dynamic model of the robot to estimate the term z_1 . Consequently, any modeling error or external disturbance will affect the contact force and the surface gradient estimation given by (23) and (24), respectively. Additionally, it is assumed that contact friction can be neglected in the mathematical model (1). This assumption might be stringent, but it is not unrealistic for some scenarios (see ref. [19]). If contact friction cannot be neglected, then it should be explicitly included both in the model (1) and the observer (16)–(21) similarly as proposed in ref. [35]. This high dependency on the model accuracy is a disadvantage that most force observers reported in the literature share, to the best of the authors' knowledge. In Section 4 we provide some experimental results to show that despite this disadvantage, our scheme was able to carry out the desired task.

3.1 Closed loop dynamics

Let $\tilde{\boldsymbol{q}}_2 \triangleq \boldsymbol{q}_2 - \hat{\boldsymbol{q}}_2$ and $\tilde{\boldsymbol{z}}_i \triangleq \boldsymbol{z}_i - \hat{\boldsymbol{z}}_i, i = 1, \dots, p$. The dynamics of the state space model (8)–(9) and (12)–(15) in closed loop with the proposed observer (16)–(21) is given by

$$\dot{\tilde{\boldsymbol{q}}}_1 = \tilde{\boldsymbol{q}}_2 - \boldsymbol{\lambda}_{p+1} \tilde{\boldsymbol{q}}_1 \tag{35}$$

$$\dot{\tilde{\boldsymbol{q}}}_2 = -\boldsymbol{H}^{-1}(\boldsymbol{q}_1) \left(\boldsymbol{N}(\boldsymbol{q}_1, \boldsymbol{q}_2) - \boldsymbol{N}(\boldsymbol{q}_1, \hat{\boldsymbol{q}}_2) \right) + \tilde{\boldsymbol{z}}_1 - \boldsymbol{\lambda}_p \tilde{\boldsymbol{q}}_1$$
(36)

$$\dot{\tilde{\boldsymbol{z}}}_1 = \tilde{\boldsymbol{z}}_2 - \boldsymbol{\lambda}_{p-1} \tilde{\boldsymbol{q}}_1 \tag{37}$$

$$\dot{\tilde{\boldsymbol{z}}}_2 = \tilde{\boldsymbol{z}}_3 - \boldsymbol{\lambda}_{p-2} \tilde{\boldsymbol{q}}_1$$
(38)

$$\dot{\tilde{\boldsymbol{z}}}_{p-1} = \tilde{\boldsymbol{z}}_p - \boldsymbol{\lambda}_1 \tilde{\boldsymbol{q}}_1 \tag{39}$$

$$\dot{\tilde{\boldsymbol{z}}}_p = \boldsymbol{r}^{(p)}(t) - \boldsymbol{\lambda}_0 \tilde{\boldsymbol{q}}_1 \,. \tag{40}$$

By the same arguments as the given in ref. [29], one can obtain

$$\ddot{\tilde{\boldsymbol{q}}}_1 + \boldsymbol{\lambda}_{p+1} \dot{\tilde{\boldsymbol{q}}}_1 + \boldsymbol{\lambda}_p \tilde{\boldsymbol{q}}_1 = \tilde{\boldsymbol{z}}_1 + \boldsymbol{f}(t), \qquad (41)$$

where

$$\boldsymbol{f}(t) \triangleq -\boldsymbol{H}^{-1}(\boldsymbol{q}_1) \left[2\boldsymbol{C}(\boldsymbol{q}_1, \dot{\boldsymbol{e}}_1 + \dot{\boldsymbol{q}}_d) (\dot{\tilde{\boldsymbol{q}}}_1 + \boldsymbol{\lambda}_{p+1} \tilde{\boldsymbol{q}}_1) - \boldsymbol{C}(\boldsymbol{q}_1, \dot{\tilde{\boldsymbol{q}}}_1 + \boldsymbol{\lambda}_{p+1} \tilde{\boldsymbol{q}}_1) (\dot{\tilde{\boldsymbol{q}}}_1 + \boldsymbol{\lambda}_{p+1} \tilde{\boldsymbol{q}}_1) + \boldsymbol{D}(\dot{\tilde{\boldsymbol{q}}}_1 + \boldsymbol{\lambda}_{p+1} \tilde{\boldsymbol{q}}_1) \right] .$$
(42)

From (37)-(40) it is easy to get

:

$$\tilde{\boldsymbol{z}}_{1}^{(p)} = \boldsymbol{r}^{(p)}(t) - \boldsymbol{\lambda}_{0} \tilde{\boldsymbol{q}}_{1} - \dots - \boldsymbol{\lambda}_{p-1} \tilde{\boldsymbol{q}}_{1}^{(p-1)}.$$
(43)

Differentiating p-times (41) and using (43) one obtains

$$\tilde{q}_{1}^{(p+2)} + \lambda_{p+1} \tilde{q}_{1}^{(p+1)} + \dots + \lambda_{0} \tilde{q}_{1} = r^{(p)}(t) + f^{(p)}(t) .$$
(44)

Equation (44) can be rewritten in a compact form as

$$\dot{\boldsymbol{x}}_{\mathrm{o}} = \boldsymbol{A}\boldsymbol{x}_{\mathrm{o}} + \boldsymbol{B}\boldsymbol{r}_{\mathrm{f}} \,, \tag{45}$$

where $\boldsymbol{r}_{\mathrm{f}} = \boldsymbol{r}^{(p)}(t) + \boldsymbol{f}^{(p)}(t)$ and

$$\boldsymbol{x}_{\mathrm{o}} \triangleq \begin{bmatrix} \tilde{\boldsymbol{q}}_{1} & \cdots & \tilde{\boldsymbol{q}}_{1}^{(p+1)} \end{bmatrix}^{\mathrm{T}}$$

$$\begin{bmatrix} \boldsymbol{O} & \boldsymbol{I} & \cdots & \boldsymbol{O} \end{bmatrix}$$
(46)

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{C} & \boldsymbol{C} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{I} \\ -\boldsymbol{\lambda}_0 & -\boldsymbol{\lambda}_1 & \cdots & -\boldsymbol{\lambda}_{p+1} \end{bmatrix}$$
(47)

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{I} \end{bmatrix}^{\mathrm{T}} .$$
 (48)

Consider for simplicity's sake $K_i = k_i I$. In order to carry out the stability analysis, the control law (29) is rewritten as

$$\boldsymbol{\tau} = -\boldsymbol{K}_{\mathrm{p}}\boldsymbol{e}_{1} - \boldsymbol{K}_{\mathrm{v}}\dot{\boldsymbol{e}}_{1} - \boldsymbol{K}_{\mathrm{i}}\boldsymbol{Q}\int_{0}^{t}\boldsymbol{e}_{1}\,\mathrm{d}\vartheta - \boldsymbol{J}_{\varphi}^{\mathrm{T}}\lambda_{\mathrm{d}} + k_{\mathrm{Fi}}\boldsymbol{J}_{\varphi}^{+}\Delta\bar{F}$$
$$+\boldsymbol{K}_{\mathrm{v}}(\dot{\boldsymbol{q}}_{1} + \boldsymbol{\lambda}_{p+1}\tilde{\boldsymbol{q}}_{1}) + \boldsymbol{K}_{\mathrm{i}}\tilde{\boldsymbol{Q}}\int_{0}^{t}\boldsymbol{e}_{1}\,\mathrm{d}\vartheta + \tilde{\boldsymbol{J}}_{\varphi}^{\mathrm{T}}\lambda_{\mathrm{d}} - k_{\mathrm{Fi}}\bar{\boldsymbol{J}}_{\varphi}^{+}\Delta\bar{F},\qquad(49)$$

where $\tilde{\boldsymbol{Q}} \triangleq \boldsymbol{Q} - \hat{\boldsymbol{Q}}, \, \tilde{\boldsymbol{J}}_{\varphi} \triangleq \boldsymbol{J}_{\varphi} - \hat{\boldsymbol{J}}_{\varphi}, \, \text{and} \, \, \bar{\boldsymbol{J}}_{\varphi}^{+} \triangleq \boldsymbol{J}_{\varphi}^{+} - \hat{\boldsymbol{J}}_{\varphi}^{+}.$ Next, it is defined

$$\dot{\boldsymbol{\sigma}} \triangleq \dot{\boldsymbol{e}}_1 + \boldsymbol{\Lambda} \boldsymbol{e}_1 \,, \tag{50}$$

where $\Lambda \in \mathbb{R}^{n \times n}$ is a diagonal positive definite gain matrix. Suppose that $Q\Lambda = \Lambda Q$ is satisfied, what can be accomplished just by setting

$$\boldsymbol{\Lambda} = k_{\lambda} \boldsymbol{I} \,, \tag{51}$$

with $k_{\lambda} > 0$. Now it is assumed momentarily that both, e_1 and \dot{e}_1 satisfy Fact 2.1 so that it is possible to find $\bar{K}_i \in \mathbb{R}^{n \times n}$ such that

$$\boldsymbol{K}_{\mathrm{p}} = \boldsymbol{K}_{\mathrm{v}} \boldsymbol{\Lambda} + \bar{\boldsymbol{K}}_{\mathrm{i}} \tag{52}$$

$$\bar{K}_{i} = \bar{K}_{i} \Lambda \,. \tag{53}$$

Hence, the control law (49) is equivalent to

$$\boldsymbol{\tau} = -\boldsymbol{K}_{\mathrm{v}}\dot{\boldsymbol{\sigma}} - \bar{\boldsymbol{K}}_{\mathrm{i}}\boldsymbol{Q}\boldsymbol{\sigma} - \boldsymbol{J}_{\varphi}^{\mathrm{T}}\lambda_{\mathrm{d}} + k_{\mathrm{Fi}}\boldsymbol{J}_{\varphi}^{+}\Delta\bar{F} + \boldsymbol{K}_{\mathrm{v}}(\dot{\boldsymbol{q}}_{1} + \boldsymbol{\lambda}_{p+1}\tilde{\boldsymbol{q}}_{1}) + \boldsymbol{K}_{\mathrm{i}}\tilde{\boldsymbol{Q}}\int_{0}^{t}\boldsymbol{e}_{1}\,\mathrm{d}\vartheta + \tilde{\boldsymbol{J}}_{\varphi}^{\mathrm{T}}\lambda_{\mathrm{d}} - k_{\mathrm{Fi}}\bar{\boldsymbol{J}}_{\varphi}^{+}\Delta\bar{F}.$$
(54)

Furthermore, it is defined

$$\dot{\boldsymbol{q}}_{\mathrm{r}} \triangleq \dot{\boldsymbol{q}}_{\mathrm{d}} - \boldsymbol{\Lambda} \boldsymbol{e}_{1} - \boldsymbol{K}_{\mathrm{v}}^{-1} \bar{\boldsymbol{K}}_{\mathrm{i}} \boldsymbol{Q} \boldsymbol{\sigma} + \frac{1}{2} \boldsymbol{K}_{\mathrm{v}}^{-1} \boldsymbol{J}_{\varphi}^{\mathrm{T}} \Delta \lambda + \frac{1}{2} k_{\mathrm{Fi}} \boldsymbol{K}_{\mathrm{v}}^{-1} \boldsymbol{J}_{\varphi}^{+} \Delta \bar{F}$$
(55)
$$\boldsymbol{s} \triangleq \dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_{\mathrm{r}} = \left(\dot{\boldsymbol{\sigma}} + \boldsymbol{K}_{\mathrm{v}}^{-1} \bar{\boldsymbol{K}}_{\mathrm{i}} \boldsymbol{Q} \boldsymbol{\sigma} \right) + \left(-\frac{1}{2} \boldsymbol{K}_{\mathrm{v}}^{-1} \boldsymbol{J}_{\varphi}^{\mathrm{T}} \Delta \lambda - \frac{1}{2} k_{\mathrm{Fi}} \boldsymbol{K}_{\mathrm{v}}^{-1} \boldsymbol{J}_{\varphi}^{+} \Delta \bar{F} \right)$$
$$\triangleq \boldsymbol{s}_{\mathrm{p}} + \boldsymbol{s}_{\mathrm{F}},$$
(56)

where $\Delta \lambda \triangleq \lambda - \lambda_d$ is the force tracking error. The closed loop dynamics (omitting the arguments for simplicity) is given by

$$\boldsymbol{H}\dot{\boldsymbol{s}} + \boldsymbol{C}\boldsymbol{s} + \boldsymbol{K}_{\mathrm{Dv}}\boldsymbol{s} = \frac{1}{2}\boldsymbol{J}_{\varphi}^{\mathrm{T}}\Delta\lambda + \frac{1}{2}k_{\mathrm{Fi}}\boldsymbol{J}_{\varphi}^{+}\Delta\bar{F} + \boldsymbol{y}_{\mathrm{a}}, \qquad (57)$$

where $\boldsymbol{K}_{\mathrm{Dv}} \triangleq \boldsymbol{K}_{\mathrm{v}} + \boldsymbol{D}$ and

$$\boldsymbol{y}_{a} = \boldsymbol{K}_{v}(\dot{\tilde{\boldsymbol{q}}}_{1} + \boldsymbol{\lambda}_{p+1}\tilde{\boldsymbol{q}}_{1}) + \boldsymbol{K}_{i}\tilde{\boldsymbol{Q}}\int_{0}^{t}\boldsymbol{e}_{1}\,\mathrm{d}\vartheta + \tilde{\boldsymbol{J}}_{\varphi}^{T}\boldsymbol{\lambda}_{d} - k_{\mathrm{F}i}\tilde{\boldsymbol{J}}_{\varphi}^{+}\Delta\bar{F} - (\boldsymbol{H}(\boldsymbol{q})\ddot{\boldsymbol{q}}_{r} + \boldsymbol{C}(\boldsymbol{q}_{1},\boldsymbol{q}_{2})\dot{\boldsymbol{q}}_{r} + \boldsymbol{D}\dot{\boldsymbol{q}}_{r} + \boldsymbol{g}(\boldsymbol{q})) .$$
(58)

Let $\tilde{J}_{\varphi x} \triangleq J_{\varphi x} - \hat{J}_{\varphi x}$ be the estimation error of the normal vector to the surface. By taking into account (24) it follows

$$\dot{\hat{\boldsymbol{J}}}_{\varphi \mathbf{x}}^{\mathrm{T}} = \dot{\boldsymbol{J}}_{\varphi \mathbf{x}}^{\mathrm{T}} - \dot{\hat{\boldsymbol{J}}}_{\varphi \mathbf{x}}^{\mathrm{T}} = \dot{\boldsymbol{J}}_{\varphi \mathbf{x}}^{\mathrm{T}} - \left(\frac{\gamma}{\hat{\lambda} + \epsilon}\right) \hat{\boldsymbol{Q}}_{\mathbf{x}} \boldsymbol{J}^{-\mathrm{T}}(\boldsymbol{q}_{1}) \boldsymbol{H}(\boldsymbol{q}_{1}) \hat{\boldsymbol{z}}_{1} \,.$$
(59)

Hence, the dynamics of system (1) in closed loop with the observer (16)–(21), the gradient estimator of surface (24) and the control law (29), is completely described by equations (33)–(34), (45), (57) and (59), for which it is defined the state vector

$$\boldsymbol{y} \triangleq \begin{bmatrix} \boldsymbol{x}_{\mathrm{o}} & \boldsymbol{s} & \Delta \bar{F} & \tilde{\boldsymbol{J}}_{\varphi \mathrm{x}}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} .$$
 (60)

Now, the main result of this work is stated as follows.

Theorem 3.1 Consider the system dynamics (1) in closed loop with the observer (16)–(21) and the controller (29), for which the complete dynamics is given by (33)–(34), (45), (57), and (59). Let y_{max} be a positive constant and let $\mathcal{D} \triangleq \{ \boldsymbol{y} \in \mathbb{R}^{(p+2)n+4} | \| \boldsymbol{y} \| \leq y_{\text{max}} \}$, where y_{max} is small enough for Fact 2.1 to hold. Assume that constraint (2) is smooth and that the manipulator tip never losses contact with the environment. Then a set of controller gains $\boldsymbol{K}_p, \boldsymbol{K}_v, \boldsymbol{K}_i$ and k_{Fi} in (29), γ in (24), and a set of observer gains $\lambda_0, \ldots, \lambda_{p+1}$ in (16)–(21) can always be found to achieve ultimate boundedness of the state \boldsymbol{y} in (60) if the initial condition $\boldsymbol{y}(t_0)$ is chosen to satisfy (112). Furthermore, the tracking errors $\boldsymbol{e}_1, \, \dot{\boldsymbol{e}}_1, \, \Delta\lambda$ and ΔF , and the estimation errors \boldsymbol{x}_0 and $\tilde{\boldsymbol{J}}_{\varphi \mathbf{x}}$ can be made arbitrarily small.

Proof 1 See Appendix A.

Remark 3.4 The gains for the observer (16)-(21), the surface estimator (24), the linear filter (27), and the hybrid force/motion controller (29) should be tuned as follows.

- 1. The poles of the observer (16)-(21) should be chosen as far in the left of the complex plane as allowed by the system bandwidth. For carrying out this tuning, any open-loop motion controller can be employed until the observation error \tilde{q}_1 is as small as possible. These gains should be gradually incremented while the observation error does not presents neither considerable peaking at the start of the transient response nor sensitivity to measurement noise.
- 2. The motion part of the force/position controller can be tuned as a common PID in free movement by forcing $\hat{Q}_x \equiv I$ and making $\lambda_d = k_{Fi} = 0$ in (29).
- 3. The gain $k_{\rm Fi}$ can be adjusted by considering a motion/force controller with some known $\hat{Q}_{\rm x}$, e.g. a flat surface.
- 4. The surface estimator gain γ in (24) should be tuned carefully as it depends on the velocity of the task and on the surface curvature. If this gain is

too low, the tip of the robot will may not be able to follow the surface contour. On the other hand, if this gain is too high, surface imperfections and measurement noise can lead to a wrong estimation and thus to a bad control performance.

5. Since (27) approximates a pure delay of one sampling period, η can be chosen simply as the inverse of the controller loop sample time, i.e. $\eta = 1/T$.

4 Simulation and experimental results

In this section a numerical simulation and a set of two experiments are presented to validate the proposed algorithm. The simulation is kept to show the good performance of the method in an ideal scenario. The experimental results, on the other hand, show the pertinence of the assumptions and the effectiveness of the approach even if there are unmodelled dynamics.

4.1 Numerical Simulation

A three–link planar robot with revolute joints in contact with a curved rigid surface was considered, as depicted in Figure 2. The parameters were taken from ref. [36] and correspond to the industrial robot CRS A465, where only joints 2, 3 and 5 are used to obtain a planar movement.



Figure 2: Three link planar robot in contact with a curved surface.

The task consisted on a trajectory over a curved surface in the plane given by the circle equation

$$(x-h)^{2} + (y-k)^{2} = r^{2}, \qquad (61)$$

where x, y are the Cartesian coordinates of the manipulator tip with respect to the base, (h, k) = (0.15, 0)[m] are the coordinates of the center of the circle and r = 0.1[m] is the corresponding radius. The initial point is (x, y) =(0.1934, 0.09)[m] from where a trajectory for the x coordinate was designed using a fifth order polynomial to achieve the final point x = 0.1[m] in a time of $t_f = 20$ [seg]. The y coordinate was calculated on-line using (28). Also, the initial error between the actual and the estimated orientation was set to 30[°].

The controller gains are $\mathbf{K}_{\rm p} = \text{diag}\{4000, 4000, 200\}, \mathbf{K}_{\rm v} = \text{diag}\{100, 100, 10\}, \mathbf{K}_{\rm i} = \text{diag}\{100, 100, 100\}, \text{ and } k_{\rm Fi} = 2$. The surface estimation gain is $\gamma = 0.6$. The degree of the Taylor polynomial in (11) is p = 2. The poles of the observer are set to $p_1 = p_2 = p_3 = p_4 = -50$, resulting in the observer gains $\lambda_0 = 6250000\mathbf{I}, \lambda_1 = 500000\mathbf{I}, \lambda_2 = 15000\mathbf{I}, \text{ and } \lambda_3 = 200\mathbf{I}, \text{ where } \mathbf{I}$ is the identity matrix of $n \times n$. The constants ϵ in (24) and η in (27) have been chosen as $\epsilon = 0.0001$ and $\eta = 500$.

The trajectory in the xy plane is shown in Figure 3, while the position tracking for the x and y coordinates and the orientation angle ϕ are shown in Figure 4. The corresponding tracking errors are displayed in Figure 5.



Figure 3: Simulation: position tracking in the xy plane, real (—), desired (- -).

In Figure 6 the desired, real and estimated forces are shown as well as the force tracking and force estimation errors. It can be seen that the force tracking and estimation is pretty accurate after the transient response. Finally, in Figure 7 the estimation of the components of $J_{\varphi x}$ are displayed. It can be appreciated that the estimated components are maintained close to the real ones during the motion of the robot and converge to them in steady state.



Figure 4: Simulation: position tracking, real (---), desired (---).

4.2 Experimental validation

The experimental setup consists of a *Geomagic Touch* 3 degrees of freedom manipulator and a planar rigid surface as shown in Figure 8. The sample time was T = 0.002[sec]. The controller gains were $\mathbf{K}_{\rm p} = \text{diag}\{2,2,2\}, \mathbf{K}_{\rm v} = \text{diag}\{0.004, 0.004, 0.004\}, \mathbf{K}_{\rm i} = \text{diag}\{0.25, 0.25, 0.25\}, \text{ and } k_{\rm Fi} = 2$. The surface estimation gain was $\gamma = 0.02$. The degree of the Taylor polynomial in (11) is p = 2. The poles of the observer are set to $p_1 = p_2 = p_3 = p_4 = -120$, resulting in the observer gains $\lambda_0 = 207360000\mathbf{I}, \lambda_1 = 6912000\mathbf{I}, \lambda_2 = 86400\mathbf{I},$ and $\lambda_3 = 480\mathbf{I}$, where \mathbf{I} is the identity matrix of $n \times n$. The constants ϵ in (24) and η in (27) have been chosen as $\epsilon = 0.0001$ and $\eta = 500$.

The experiment consisted on following a circle over an horizontal flat surface. The desired contact force as well as the x and y coordinates were planned offline by means of a fifth degree polynomial in time, while the z coordinate was planned online to fulfill (28). The initial condition proposed for the gradient was $\hat{J}_{\varphi x}(t_0) = \begin{bmatrix} \sqrt{0.1} & -\sqrt{0.1} & \sqrt{0.8} \end{bmatrix}$. It is considered that the real gradient is $J_{\varphi x}(t) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$.

A projection of the trajectory in the xy plane is shown in Figure 9. The position tracking in Cartesian coordinates is displayed in Figure 10, while the corresponding tracking errors are shown in Figure 11. Note that all the errors remain bounded and that the ultimate bound for these errors is close to zero.

In Figure 12 the normal component of the desired, measured and estimated contact forces is shown, as well as the corresponding force tracking error and force estimation error. Note that the errors remain bounded during all the experiment and are close to zero in steady state.

The evolution of the estimated gradient components is shown in Figure 13



Figure 5: Simulation: position tracking errors.

along with the expected values. Moreover, the (unitary) estimated gradient vector at some fixed times is displayed in Figure 14. Although the estimated vector does never coincide exactly with the expected one, mainly due to the effect of the unmodeled dynamics, it evolves to a better estimation than the proposed initial condition.



Figure 6: Simulation: a) Force tracking (normal component), real (—), desired (- -), estimated (\cdots), b) force tracking error and c) force estimation error.

4.3 Discussion

As it can be seen in the last two sections, the experimental results are not longer as accurate in performance as the simulation results. The main causes of this performance losing are the uncertainties in the mathematical model and the external disturbances. Notice that any external disturbance enters in the model (1) in the same manner as the contact forces vector. Since the observer (16)-(21)relies on the cancellation of all the manipulator dynamics terms, it can be seen



Figure 7: Simulation: estimation of $J_{\varphi x}$, real (—), estimated (- -).



Figure 8: Experimental setup.

that any external disturbance is undistinguishable from the contact force. On the other hand, the uncertainties on the Coriolis, damping and gravity forces terms affects in the same way the force estimation as the external disturbances.



Figure 9: Experiment: position tracking in the xy plane, real (—), desired (- -).



Figure 10: Experiment: position tracking time evolution, real (—), desired (- -).

Nevertheless, in this case some of the well-known properties for robotic manipulators could be employed to eliminate or at least reduce the dependence on the accuracy of the model, which has not been done yet in the literature, to the best of the authors' knowledge. The inertia matrix uncertainties have also effect on the force estimation since this matrix is accounted in (23). The only difference is that the uncertainties enter in an homogeneous rather than in an



Figure 11: Experiment: position tracking errors.



Figure 12: Experiment: a) Force tracking (normal component), real (—), desired (- -), estimated (\cdots), b) Force tracking error, c) Force estimation error.

additive form.

The same conclusions as above apply for the unknown surface estimator as it can be seen as the direction of the unknown force vector, while the contact force can be seen as the magnitude of the same vector.



Figure 13: Experiment: estimation of $J_{\varphi x}$, estimated (—), actual (- - -)



Figure 14: Experiment: Estimation of $\boldsymbol{J}_{\varphi\mathbf{x}}$ at different times.

5 Conclusions

In this work, under the assumption that neither the contact force nor the joint velocities are measurable, an extended state high–gain observer was designed to solve the problem of motion and force tracking over a geometrically uncertain surface. The proposed observer–based control scheme guarantees ultimate boundedness of position, velocity observation and force tracking errors with an arbitrary small ultimate bound. Besides, the algorithm estimates on line the normal vector to the surface to correct the desired position and orientation of the manipulator tip, which is equivalent to locally estimate the unknown contact surface while moving over it. The simulation results show that the proposed method is capable of accurately estimate the surface gradient, the joint velocity and the contact force and at the same time while following a trajectory in position and force. On the other hand, in the experiments the tracking and the estimation is no longer as accurate as in the simulations. This can be due to the presence of unmodelled dynamics, such as contact friction, and disturbances. Nevertheless, all the tracking errors remained bounded with ultimate bound close to the origin of the error dynamics as guaranteed by the mathematical analysis.

As future work, a model of contact friction will be included to improve the performance in the experimental results. Also, it will be investigated the application of the proposed force/velocity observer for the force control problem over deformable surfaces.

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A Proof of Theorem 3.1

As stated in Theorem 3.1, the stability result is local and valid only in a region \mathcal{D} of interest, where it is assumed that Fact 2.1 holds. Therefore, it must be shown that any signal of interest is bounded whenever $\boldsymbol{y}_{\mathrm{s}} \in \mathcal{D}$ and that, with a proper choice of gains, $\boldsymbol{y}_{\mathrm{s}}$ will stay in \mathcal{D} for all time and will tend to an arbitrary small region. Consider the next four steps:

a) First, we show that whenever the state \boldsymbol{y}_{s} in (60) is bounded *i.e.*, $\boldsymbol{y} \in \mathcal{D}$, every signal of interest is also bounded. From (56) it is

$$\dot{\boldsymbol{\sigma}} = -\boldsymbol{K}_{\mathrm{v}}^{-1} \bar{\boldsymbol{K}}_{\mathrm{i}} \boldsymbol{Q} \boldsymbol{\sigma} + \boldsymbol{s}_{\mathrm{p}} \,, \tag{62}$$

where \mathbf{s}_{p} is bounded in \mathcal{D} , because due to Fact 2.1 \mathbf{s}_{p} and \mathbf{s}_{F} are orthogonal. For simplicity's sake consider $\mathbf{K}_{\mathrm{v}} = k_{\mathrm{v}}\mathbf{I}$ and $\bar{\mathbf{K}}_{\mathrm{i}} = \bar{k}_{\mathrm{i}}\mathbf{I}$. Then it can be shown that $\boldsymbol{\sigma}$ and $\dot{\boldsymbol{\sigma}}$ are bounded and $\|\boldsymbol{Q}\boldsymbol{\sigma}\|$ and $\|\dot{\boldsymbol{\sigma}}\|$ can be made arbitrarily small by setting \bar{k}_{i} large enough (see ref. [29] for details). As a result, from (50) $\dot{\boldsymbol{e}}_{1}$, \boldsymbol{e}_{1} , and $\int_{0}^{t} \boldsymbol{e}_{1} \, d\vartheta$ are bounded and, since $\dot{\boldsymbol{q}}_{\mathrm{d}}$ and $\boldsymbol{q}_{\mathrm{d}}$ are bounded as well. Furthermore, $\boldsymbol{q}_{1}, \boldsymbol{q}_{2}, \hat{\boldsymbol{q}}_{1}, \dot{\boldsymbol{q}}_{1}$, and $\hat{\boldsymbol{q}}_{2}$ are also bounded after (35) and because $\boldsymbol{x}_{\mathrm{o}}$ is bounded in \mathcal{D} . From (25) and (30)–(32) it follows that $\hat{\boldsymbol{J}}_{\varphi \mathrm{x}}, \, \hat{\boldsymbol{J}}_{\varphi}, \, \hat{\boldsymbol{Q}}_{\mathrm{x}}$, and $\hat{\boldsymbol{Q}}$ are bounded. This implies after (29) that $\boldsymbol{\tau}$ is bounded. Now, consider³⁷

$$\boldsymbol{\lambda} = \left(\boldsymbol{J}_{\varphi}(\boldsymbol{q}_{1}) \boldsymbol{H}^{-1}(\boldsymbol{q}_{1}) \boldsymbol{J}_{\varphi}^{\mathrm{T}}(\boldsymbol{q}_{1}) \right)^{-1} \left\{ \boldsymbol{J}_{\varphi}(\boldsymbol{q}_{1}) \boldsymbol{H}^{-1}(\boldsymbol{q}_{1}) \left(\boldsymbol{\tau} - \boldsymbol{N}(\boldsymbol{q}_{1}, \boldsymbol{q}_{2}) \right) + \boldsymbol{J}_{\varphi}(\boldsymbol{q}_{1}, \boldsymbol{q}_{2}) \boldsymbol{q}_{2} \right\}.$$
(63)

After Property 2.1, λ is bounded, which in turn means that z_1 in (10) is bounded too. By taking into account Assumption 2.1 and that the

surface is smooth, the partial derivatives $\partial \varphi(\mathbf{q}_1)/\partial \mathbf{q}_1, \partial^2 \varphi(\mathbf{q}_1)/\partial \mathbf{q}_1^2, \ldots, \partial^{p+1} \varphi(\mathbf{q}_1)/\partial \mathbf{q}_1^{p+1}$ are bounded. Therefore, $\dot{\mathbf{J}}_{\varphi}(\mathbf{q}_1) = (\partial \mathbf{J}_{\varphi}(\mathbf{q}_1)/\partial \mathbf{q}_1) \dot{\mathbf{q}}_1$ must be bounded as well as $\mathbf{f}(t)$ in (42) and $\dot{\mathbf{q}}_r$ in (55). From (41) one can conclude that $\tilde{\mathbf{z}}_1$ is bounded and, as a consequence, $\hat{\mathbf{z}}_1$ and $\hat{\lambda}$ in (23), and $\Delta \bar{\lambda}$ in (33) must be bounded. After (24), (32), and (59), $\dot{\mathbf{J}}_{\varphi}, \dot{\mathbf{J}}_{\varphi}$, and $\dot{\mathbf{Q}}$ are bounded as well.

Taking into account (63) and the manipulator model (8)–(9), one can write the joint acceleration as a function of only (q_1, q_2, τ) , *i.e.*,

$$\dot{\boldsymbol{q}}_2 = \ddot{\boldsymbol{q}}_1 = \boldsymbol{f}_q(\boldsymbol{q}_1, \dot{\boldsymbol{q}}_1, \boldsymbol{\tau}), \qquad (64)$$

which clearly shows that $\dot{\boldsymbol{q}}_2$ is bounded. Since $\ddot{\boldsymbol{q}}_d$ and $\ddot{\tilde{\boldsymbol{q}}}_1$ are bounded, $\ddot{\boldsymbol{e}}_1$, $\ddot{\tilde{\boldsymbol{q}}}_1$ and after (35) $\dot{\tilde{\boldsymbol{q}}}_2$ must be bounded as well. Now, by similar arguments, $\boldsymbol{\tau}$ in (29) can be written as a function of bounded variables, *i.e.*,

$$\boldsymbol{\tau} = \boldsymbol{f}_{\tau} \left(\dot{\boldsymbol{q}}_{\mathrm{d}}, \int_{0}^{t} \boldsymbol{e}_{1} \,\mathrm{d}\vartheta, \boldsymbol{e}_{1}, \tilde{\boldsymbol{q}}_{1}, \dot{\tilde{\boldsymbol{q}}}_{1}, \hat{\boldsymbol{Q}}, \lambda_{\mathrm{d}}, \Delta \bar{F} \right) \,. \tag{65}$$

Therefore, its time derivative must be a function of the form

$$\dot{\boldsymbol{\tau}} = \dot{\boldsymbol{f}}_{\tau} \left(\dot{\boldsymbol{q}}_{\mathrm{d}}, \ddot{\boldsymbol{q}}_{\mathrm{d}}, \int_{0}^{t} \boldsymbol{e}_{1} \,\mathrm{d}\vartheta, \boldsymbol{e}_{1}, \dot{\boldsymbol{e}}_{1}, \ddot{\boldsymbol{q}}_{1}, \dot{\tilde{\boldsymbol{q}}}_{1}, \ddot{\tilde{\boldsymbol{q}}}_{1}, \dot{\tilde{\boldsymbol{Q}}}, \dot{\hat{\boldsymbol{Q}}}, \lambda_{\mathrm{d}}, \dot{\lambda}_{\mathrm{d}}, \Delta \bar{F}, \Delta \bar{\lambda} \right), \quad (66)$$

which is bounded since it depends on variables we have already proven to be bounded. The time derivative of \dot{q}_r is given by

$$\ddot{\boldsymbol{q}}_{\mathrm{r}} = \ddot{\boldsymbol{q}}_{\mathrm{d}} - \boldsymbol{\Lambda} \dot{\boldsymbol{e}}_{1} - \boldsymbol{K}_{\mathrm{v}}^{-1} \bar{\boldsymbol{K}}_{\mathrm{i}} \dot{\boldsymbol{Q}} \boldsymbol{\sigma} - \boldsymbol{K}_{\mathrm{v}}^{-1} \bar{\boldsymbol{K}}_{\mathrm{i}} \boldsymbol{Q} \dot{\boldsymbol{\sigma}} + \frac{1}{2} \boldsymbol{K}_{\mathrm{v}}^{-1} \dot{\boldsymbol{J}}_{\varphi}^{\mathrm{T}} \Delta \lambda + \frac{1}{2} \boldsymbol{K}_{\mathrm{v}}^{-1} \boldsymbol{J}_{\varphi}^{\mathrm{T}} \frac{\mathrm{d}}{\mathrm{d}t} (\Delta \lambda) + \frac{1}{2} k_{\mathrm{Fi}} \boldsymbol{K}_{\mathrm{v}}^{-1} \dot{\boldsymbol{J}}_{\varphi}^{+} \Delta \bar{\boldsymbol{F}} + \frac{1}{2} k_{\mathrm{Fi}} \boldsymbol{K}_{\mathrm{v}}^{-1} \boldsymbol{J}_{\varphi}^{+} \Delta \bar{\lambda} , \quad (67)$$

which again turns out to be bounded since from (63), $\frac{d}{dt}(\Delta\lambda)$ is a function only of $(\boldsymbol{q}_1, \dot{\boldsymbol{q}}_1, \ddot{\boldsymbol{q}}_1, \boldsymbol{\tau}, \dot{\boldsymbol{\tau}})$. Then, there must exist a positive constant c_a such that \boldsymbol{y}_a in (58) fulfils $\|\boldsymbol{y}_a\| \leq c_a$, whenever $\boldsymbol{y} \in \mathcal{D}$. As a direct consequence, $\dot{\boldsymbol{s}}$ in (57) is bounded. By differentiating (56) one obtains

$$\dot{\boldsymbol{s}} = \ddot{\boldsymbol{\sigma}} + \boldsymbol{K}_{v}^{-1} \bar{\boldsymbol{K}}_{i} \dot{\boldsymbol{Q}} \boldsymbol{\sigma} + \boldsymbol{K}_{v}^{-1} \bar{\boldsymbol{K}}_{i} \boldsymbol{Q} \dot{\boldsymbol{\sigma}} - \frac{1}{2} \boldsymbol{K}_{v}^{-1} \dot{\boldsymbol{J}}_{\varphi}^{\mathrm{T}} \Delta \lambda - \frac{1}{2} \boldsymbol{K}_{v}^{-1} \boldsymbol{J}_{\varphi}^{\mathrm{T}} \frac{\mathrm{d}}{\mathrm{d}t} (\Delta \lambda) - \frac{1}{2} k_{\mathrm{Fi}} \boldsymbol{K}_{v}^{-1} \dot{\boldsymbol{J}}_{\varphi}^{+} \Delta \bar{F} - \frac{1}{2} k_{\mathrm{Fi}} \boldsymbol{K}_{v}^{-1} \boldsymbol{J}_{\varphi}^{+} \Delta \bar{\lambda}, \qquad (68)$$

so that $\ddot{\sigma}$ must be bounded.

At this point, an iterative argument is carried out. First, by computing the time derivative of (64) it is

$$\boldsymbol{q}_{1}^{(3)} = \dot{\boldsymbol{f}}_{q}(\boldsymbol{q}_{1}, \dot{\boldsymbol{q}}_{1}, \ddot{\boldsymbol{q}}_{1}, \boldsymbol{\tau}, \dot{\boldsymbol{\tau}}), \qquad (69)$$

which shows that $q_1^{(3)}$, $e_1^{(3)}$ and $\hat{q}_1^{(3)}$ are bounded. Combining (16) and (17), it can be written

$$\hat{z}_1 = f_{\hat{z}_1}(q_1, \hat{q}_1, \hat{q}_1, \dot{q}_1, \dot{q}_2, \tau).$$
 (70)

This implies that

$$\dot{\hat{z}}_{1} = \dot{f}_{\hat{z}_{1}} \left(q_{1}, \dot{q}_{1}, \ddot{\hat{q}}_{1}, \dot{\hat{q}}_{1}, \dot{\hat{q}}_{1}, \ddot{\hat{q}}_{1}, \ddot{\hat{q}}_{1}, \ddot{\hat{q}}_{1}, \dot{\hat{q}}_{2}, \dot{\hat{q}}_{2}, \tau, \dot{\tau} \right)$$
(71)

is bounded and so are $\dot{\hat{\lambda}}$ and $\frac{d}{dt}(\Delta \bar{\lambda})$ as a consequence. From (24) it is

$$\hat{\boldsymbol{J}}_{\varphi \mathbf{x}} = \dot{\boldsymbol{f}}_{\hat{\mathbf{J}}_{\varphi \mathbf{x}}}(\boldsymbol{q}_1, \hat{\boldsymbol{z}}_1), \qquad (72)$$

where (23) has been taken into account. Then, after (30) it means

$$\hat{\boldsymbol{J}}_{\varphi} = \dot{\boldsymbol{f}}_{\hat{\boldsymbol{J}}_{\varphi}}(\boldsymbol{q}_{1}, \dot{\boldsymbol{q}}_{1}, \hat{\boldsymbol{z}}_{1})$$
(73)

$$\hat{\boldsymbol{J}}_{\varphi} = \ddot{\boldsymbol{f}}_{\hat{\boldsymbol{J}}_{\varphi}}(\boldsymbol{q}_1, \dot{\boldsymbol{q}}_1, \ddot{\boldsymbol{q}}_1, \dot{\boldsymbol{z}}_1, \dot{\boldsymbol{z}}_1), \qquad (74)$$

which implies that $\ddot{\hat{Q}}$ is bounded. On the other hand

$$\ddot{\boldsymbol{J}}_{\varphi} = \boldsymbol{f}_{\mathrm{J}_{\varphi}}(\boldsymbol{q}_{1}, \dot{\boldsymbol{q}}_{1}, \ddot{\boldsymbol{q}}_{1}) \tag{75}$$

must be bounded from the assumptions on the smoothness of the surface and because \dot{q}_1 and \dot{q}_2 are bounded, which along with (74) means that $\ddot{\tilde{J}}_{\varphi}$ is bounded. Now, from (66) it is

$$\ddot{\boldsymbol{\tau}} = \ddot{\boldsymbol{f}}_{\tau} \left(\dot{\boldsymbol{q}}_{\mathrm{d}}, \dots, \boldsymbol{q}_{\mathrm{d}}^{(3)}, \int_{0}^{t} \boldsymbol{e}_{1} \,\mathrm{d}\vartheta, \boldsymbol{e}_{1}, \dots, \ddot{\boldsymbol{e}}_{1}, \tilde{\boldsymbol{q}}_{1}, \dots, \tilde{\boldsymbol{q}}_{1}^{(3)}, \\ \hat{\boldsymbol{Q}}, \dots, \ddot{\boldsymbol{Q}}, \lambda_{\mathrm{d}}, \dots, \ddot{\lambda}_{\mathrm{d}}, \Delta \bar{F}, \Delta \bar{\lambda}, \frac{\mathrm{d}}{\mathrm{d}t} (\Delta \bar{\lambda}) \right), \quad (76)$$

which is bounded from the same arguments as those in the previous discussion. By the definition of f in (42) it can be written

$$\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{q}_1, \dot{\boldsymbol{q}}_1, \dot{\boldsymbol{q}}_1, \dot{\tilde{\boldsymbol{q}}}_1, \dot{\tilde{\boldsymbol{q}}}_1)$$
(77)

$$\dot{\boldsymbol{f}} = \dot{\boldsymbol{f}}(\boldsymbol{q}_1, \dot{\boldsymbol{q}}_1, \ddot{\boldsymbol{q}}_1, \dot{\boldsymbol{q}}_1, \ddot{\boldsymbol{q}}_1, \ddot{\boldsymbol{q}}_1, \ddot{\boldsymbol{q}}_1, \ddot{\boldsymbol{q}}_1, \ddot{\boldsymbol{q}}_1).$$
(78)

As a result, from (41) one can conclude that $\dot{\tilde{z}}_1$ is bounded, and so is \dot{z}_1 . Following this procedure iteratively, it is obtained

$$\boldsymbol{q}_{1}^{(p+1)} = \boldsymbol{f}_{q}^{(p-1)} \left(\boldsymbol{q}_{1}, \dots, \boldsymbol{q}_{1}^{(p)}, \boldsymbol{\tau}, \dots, \boldsymbol{\tau}^{(p-1)} \right),$$
(79)

which means that $q_1^{(p+1)}$ and $\hat{q}_1^{(p+1)}$ (and all their previous derivatives) are bounded. From (71) it follows

$$\hat{\boldsymbol{z}}_{1}^{(p-1)} = \boldsymbol{f}_{\hat{\boldsymbol{z}}_{1}}^{(p-1)} \left(\boldsymbol{q}_{1}, \dots, \boldsymbol{q}_{1}^{(p-1)}, \hat{\boldsymbol{q}}_{1}, \dots, \hat{\boldsymbol{q}}_{1}^{(p+1)}, \tilde{\boldsymbol{q}}_{1}, \dots, \tilde{\boldsymbol{q}}_{1}^{(p)}, \\ \dot{\hat{\boldsymbol{q}}}_{2}, \dots \hat{\boldsymbol{q}}_{2}^{(p-1)}, \boldsymbol{\tau}, \dots, \boldsymbol{\tau}^{(p-1)} \right), \quad (80)$$

which must be bounded as well as all its previous time derivatives. It also implies that $\ddot{\lambda}, \ldots, \hat{\lambda}^{(p-1)}$ and $d^2(\Delta \bar{\lambda})/dt^2, \ldots, d^{p-1}(\Delta \bar{\lambda})/dt^{p-1}$ are bounded. In the same manner, from (74) it is

$$\hat{\boldsymbol{J}}_{\varphi}^{(p)} = \boldsymbol{f}_{\hat{J}_{\varphi}}^{(p)} \left(\boldsymbol{q}_{1}, \dots, \boldsymbol{q}_{1}^{(p)}, \hat{\boldsymbol{z}}_{1}, \dots, \hat{\boldsymbol{z}}_{1}^{(p-1)} \right) .$$
(81)

Also, from (76) it is obtained

$$\boldsymbol{\tau}^{(p)} = \boldsymbol{f}_{\tau}^{(p)} \left(\dot{\boldsymbol{q}}_{\mathrm{d}}, \dots, \boldsymbol{q}_{\mathrm{d}}^{(p+1)}, \int_{0}^{t} \boldsymbol{e}_{1} \,\mathrm{d}\vartheta, \boldsymbol{e}_{1}, \dots, \boldsymbol{e}_{1}^{(p)}, \tilde{\boldsymbol{q}}_{1}, \dots, \tilde{\boldsymbol{q}}_{1}^{(p+1)}, \\ \hat{\boldsymbol{Q}}, \dots, \hat{\boldsymbol{Q}}^{(p)}, \lambda_{\mathrm{d}}, \dots, \lambda_{\mathrm{d}}^{(p)}, \Delta \bar{F}, \Delta \bar{\lambda}, \dots, \mathrm{d}^{(p-1)}(\Delta \bar{\lambda})/\mathrm{d}t^{(p-1)} \right),$$

that is bounded, since it is a function of bounded signals. On the other hand, from (78) it can be written

$$\boldsymbol{f}^{(p)} = \boldsymbol{f}^{(p)} \left(\boldsymbol{q}_1, \dots, \boldsymbol{q}_1^{(p+1)}, \dot{\boldsymbol{q}}_d, \dots, \boldsymbol{q}_d^{(p+1)}, \tilde{\boldsymbol{q}}_1, \dots, \tilde{\boldsymbol{q}}_1^{(p+1)} \right), \quad (82)$$

which is bounded with all its previous derivatives bounded too. From (41) it is computed

$$\tilde{\boldsymbol{q}}_{1}^{(p+1)} + \boldsymbol{\lambda}_{p+1} \tilde{\boldsymbol{q}}_{1}^{(p)} + \boldsymbol{\lambda}_{p} \tilde{\boldsymbol{q}}_{1}^{(p-1)} = \tilde{\boldsymbol{z}}_{1}^{(p-1)} + \boldsymbol{f}^{(p-1)}, \qquad (83)$$

which implies that $(\tilde{z}_1, \ldots, \tilde{z}_1^{(p-1)})$ are bounded too. Also, from (10) and (63) it can be seen that

$$\boldsymbol{z}_1 = \boldsymbol{f}_{\mathbf{z}_1}(\boldsymbol{q}_1, \dot{\boldsymbol{q}}_1, \boldsymbol{\tau}), \qquad (84)$$

from where it can be stated that all its time derivatives up to

$$\boldsymbol{z}_{1}^{(p)} = \boldsymbol{f}_{z_{1}}^{(p)} \left(\boldsymbol{q}_{1}, \dots, \boldsymbol{q}_{1}^{(p+1)}, \boldsymbol{\tau}, \dots, \boldsymbol{\tau}^{(p)} \right)$$
(85)

must be bounded as well. From (12)–(15) it can be concluded that $(\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_p, \dot{\boldsymbol{z}}_1, \dot{\boldsymbol{z}}_2, \ldots, \dot{\boldsymbol{z}}_p)$ and $\boldsymbol{r}^{(p)}$ must be bounded, since $\boldsymbol{z}_p = \boldsymbol{z}_1^{(p)}$. Furthermore, from (21), one can see that $\dot{\boldsymbol{z}}_p$ is bounded as well. Also, since $(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_1^{(p-1)})$ are bounded, one can easily show that the estimated variables $(\boldsymbol{z}_2, \ldots, \boldsymbol{z}_p, \dot{\boldsymbol{z}}_2, \ldots, \dot{\boldsymbol{z}}_{p-1})$ must be bounded. Moreover, all the related errors must be bounded as well.

b) The second step of the proof is completely analogous to that given in ref. [29] from which only the main points are recalled. Let

$$V_{\rm a} = \boldsymbol{x}_{\rm o}^{\rm T} \boldsymbol{P}_{\rm o} \boldsymbol{x}_{\rm o} \,, \tag{86}$$

with \boldsymbol{x}_{o} defined in (46) and $\boldsymbol{P}_{o} = \boldsymbol{P}_{o}^{T} > \boldsymbol{O}$ given as the solution of

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}_{\mathrm{o}} + \boldsymbol{P}_{\mathrm{o}}\boldsymbol{A} = -\boldsymbol{Q}_{\mathrm{o}}, \qquad (87)$$

where \boldsymbol{Q}_{o} is a positive definite matrix and \boldsymbol{A} is given by (47). Whenever $\boldsymbol{y} \in \mathcal{D}, \boldsymbol{r}_{f}$ in (45) is bounded and there must exist a constant, say r_{\max} , such that $\sup_{t_{0} < \vartheta < t} \|\boldsymbol{r}_{f}(\vartheta)\| \leq r_{\max}$. The time derivative of V_{a} fulfils

$$\dot{V}_{\mathrm{a}} \leq -\|\boldsymbol{x}_{\mathrm{o}}\| \left(\lambda_{\mathrm{min}}(\boldsymbol{Q}_{\mathrm{o}})\|\boldsymbol{x}_{\mathrm{o}}\| - 2\lambda_{\mathrm{max}}(\boldsymbol{P}_{\mathrm{o}})\|\boldsymbol{B}\|r_{\mathrm{max}}\right) \,. \tag{88}$$

Then, it follows

$$\dot{V}_{a} \leq 0 \quad \text{if} \quad \|\boldsymbol{x}_{o}(t)\| \geq \frac{2\lambda_{\max}(\boldsymbol{P}_{o})}{\lambda_{\min}(\boldsymbol{Q}_{o})} \|\boldsymbol{B}\| r_{\max},$$
(89)

where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the minimum and the maximum eigenvalue of their arguments, respectively. Also, since the system (45) is linear, by properly choosing the eigenvalues of \boldsymbol{A} , there must exist an ultimate bound for \boldsymbol{x}_{o} given by³⁸

$$\|\boldsymbol{x}_{\mathrm{o}}(t)\| \leq \frac{n(p+1)}{|\lambda_{\mathrm{max}}(\boldsymbol{A})|} \|\boldsymbol{B}\| r_{\mathrm{max}}, \quad \text{as} \quad t \to \infty.$$
(90)

Since n(p+1) is fixed and $|\lambda_{\max}(\mathbf{A})|$ can be chosen arbitrarily large, the ultimate bound of \mathbf{x}_{o} can be made arbitrarily small. Besides, it can be proved that the ultimate bound of \mathbf{x}_{o} also fulfils³⁹

$$\|\boldsymbol{x}_{\mathrm{o}}(t)\| \leq \frac{2\lambda_{\mathrm{max}}(\boldsymbol{P}_{\mathrm{o}})}{\lambda_{\mathrm{min}}(\boldsymbol{Q}_{\mathrm{o}})} \sqrt{\frac{\lambda_{\mathrm{max}}(\boldsymbol{P}_{\mathrm{o}})}{\lambda_{\mathrm{min}}(\boldsymbol{P}_{\mathrm{o}})}} \|\boldsymbol{B}\| r_{\mathrm{max}}, \qquad (91)$$

which means (*c.f.* (90)), that the term $\lambda_{\max}(\boldsymbol{P}_{o})/\lambda_{\min}(\boldsymbol{Q}_{o})$ can be made arbitrarily small. Notice that the ultimate bound of $\|\boldsymbol{x}_{o}\|$ in (90) can be made arbitrarily small independently of the norm of the state \boldsymbol{y} in (60), *i.e.*, it depends only on the choose of the eigenvalues of \boldsymbol{A} in (47).

c) Till now it has been shown that whenever $\boldsymbol{y} \in \mathcal{D}$, every signal of interest is bounded and furthermore, the observation errors can be made arbitrarily small. The next step is to show that whenever $\boldsymbol{y}(t_0)$ is small enough, then it can be enforced for \boldsymbol{y} to remain in \mathcal{D} and that actually $\|\boldsymbol{y}\|$ can be made arbitrarily small, *i.e.*, \boldsymbol{y} is ultimately bounded with ultimate bound arbitrarily small (see Fig A1 of ref. [29] for details).

Let

$$V_{\rm s} = \frac{1}{2} \boldsymbol{s}^{\rm T} \boldsymbol{H}(\boldsymbol{q}) \boldsymbol{s} + \frac{1}{4} \frac{k_{\rm Fi}}{k_{\rm v}} (\Delta \bar{F})^2$$
(92)

be a positive function of s and $\Delta \bar{F}$.

From Properties 2.1-2.3, it can be shown that the time derivative of (92) along (34) and (57) is given by

$$\begin{split} \dot{V}_{s} &= -\boldsymbol{s}^{\mathrm{T}}\boldsymbol{K}_{\mathrm{vD}}\boldsymbol{s} + \boldsymbol{s}^{\mathrm{T}}\boldsymbol{y}_{\mathrm{a}} - \frac{1}{4}k_{\mathrm{v}}^{-1}(\Delta\lambda)^{2}\boldsymbol{J}_{\varphi}\boldsymbol{J}_{\varphi}^{\mathrm{T}} - \frac{1}{2}\frac{k_{\mathrm{Fi}}}{k_{\mathrm{v}}}\Delta\bar{F}(\boldsymbol{J}_{\varphi}^{+})^{\mathrm{T}}\boldsymbol{H}(\boldsymbol{q})\tilde{\boldsymbol{z}}_{1} \\ &- \frac{1}{4}\frac{k_{\mathrm{Fi}}^{2}}{k_{\mathrm{v}}}(\Delta\bar{F})^{2}(\boldsymbol{J}_{\varphi}\boldsymbol{J}_{\varphi}^{\mathrm{T}})^{-1} \\ &\leq -k_{\mathrm{v}}\|\boldsymbol{s}\|^{2} + c_{\mathrm{a}}\|\boldsymbol{s}\| + \frac{1}{2}\frac{k_{\mathrm{Fi}}\lambda_{\mathrm{H}}c_{\varphi}^{+}}{k_{\mathrm{v}}}|\Delta\bar{F}|\|\tilde{\boldsymbol{z}}_{1}\| - \frac{1}{4}\frac{k_{\mathrm{Fi}}^{2}c_{\varphi}^{-}}{k_{\mathrm{v}}}|\Delta\bar{F}|^{2}, \quad (93) \end{split}$$



Figure 15: Region \mathcal{S}

where $c_{\varphi}^{+} \triangleq \|J_{\varphi}^{+}\|_{\max}$ and $c_{\varphi}^{-} \triangleq \inf_{\boldsymbol{q} \in \mathbb{R}^{n}} \left\{ J_{\varphi}(\boldsymbol{q}) J_{\varphi}^{\mathrm{T}}(\boldsymbol{q}) \right\}^{-1}$. Note that, since $J_{\varphi}(\boldsymbol{q})$ is full rank for every $\boldsymbol{q} \in \mathbb{R}^{n}$ and the Jacobian of the manipulator is nonsingular and upper bounded, it is $0 < c_{\varphi}^{\pm} < \infty$. On the other hand, define

$$V_{\varphi \mathbf{x}} = \frac{1}{2} \tilde{\boldsymbol{J}}_{\varphi \mathbf{x}} \tilde{\boldsymbol{J}}_{\varphi \mathbf{x}}^{\mathrm{T}} \,. \tag{94}$$

Its time derivative is given by

$$\dot{V}_{\varphi \mathbf{x}} = \tilde{\boldsymbol{J}}_{\varphi \mathbf{x}} \dot{\tilde{\boldsymbol{J}}}_{\varphi \mathbf{x}}^{\mathrm{T}} = -\tilde{\boldsymbol{J}}_{\varphi \mathbf{x}} \dot{\tilde{\boldsymbol{J}}}_{\varphi \mathbf{x}}^{\mathrm{T}} + \tilde{\boldsymbol{J}}_{\varphi \mathbf{x}} \dot{\boldsymbol{J}}_{\varphi \mathbf{x}}^{\mathrm{T}}.$$
(95)

After (22) and (24), and since $\tilde{z}_1 = z_1 - \hat{z}_1$, one obtains

$$\dot{V}_{\varphi \mathbf{x}} = -\frac{\gamma}{\hat{\lambda} + \epsilon} \tilde{\boldsymbol{J}}_{\varphi \mathbf{x}} \hat{\boldsymbol{Q}}_{\mathbf{x}} \left(\boldsymbol{J}_{\varphi \mathbf{x}}^{\mathrm{T}} \lambda - \boldsymbol{J}^{-\mathrm{T}}(\boldsymbol{q}_{1}) \boldsymbol{H}(\boldsymbol{q}_{1}) \tilde{\boldsymbol{z}}_{1} \right) + \tilde{\boldsymbol{J}}_{\varphi \mathbf{x}} \dot{\boldsymbol{J}}_{\varphi \mathbf{x}}^{\mathrm{T}}.$$
 (96)

Hence, because $\hat{\boldsymbol{Q}}_{\mathrm{x}}\hat{\boldsymbol{J}}_{\varphi\mathrm{x}}^{\mathrm{T}} = \boldsymbol{O}$ and $\hat{\boldsymbol{Q}}_{\mathrm{x}}\hat{\boldsymbol{Q}}_{\mathrm{x}} = \hat{\boldsymbol{Q}}_{\mathrm{x}} = \hat{\boldsymbol{Q}}_{\mathrm{x}}^{\mathrm{T}}$, it is

$$\dot{V}_{\varphi \mathbf{x}} = -\frac{\gamma}{\hat{\lambda} + \epsilon} \left(\tilde{\boldsymbol{J}}_{\varphi \mathbf{x}} \hat{\boldsymbol{Q}}_{\mathbf{x}}^{\mathrm{T}} \hat{\boldsymbol{Q}}_{\mathbf{x}} \tilde{\boldsymbol{J}}_{\varphi \mathbf{x}}^{\mathrm{T}} \lambda - \tilde{\boldsymbol{J}}_{\varphi \mathbf{x}} \hat{\boldsymbol{Q}}_{\mathbf{x}} \boldsymbol{J}^{-\mathrm{T}}(\boldsymbol{q}_{1}) \boldsymbol{H}(\boldsymbol{q}_{1}) \tilde{\boldsymbol{z}}_{1} \right) + \tilde{\boldsymbol{J}}_{\varphi \mathbf{x}} \dot{\boldsymbol{J}}_{\varphi \mathbf{x}}^{\mathrm{T}} \,.$$
(97)

Since \boldsymbol{q}_1 and $\dot{\boldsymbol{q}}_1$ are bounded in \mathcal{D} and the surface is assumed to be smooth, there must exist a positive constant, say v_x , such that $\|\dot{\boldsymbol{J}}_{\varphi x}\| \leq v_x < \infty$. Also, consider a closed subset of the workspace of the manipulator centred at $\{\boldsymbol{x} \in \mathbb{R}^n | \tilde{\boldsymbol{J}}_{\varphi x}(\boldsymbol{x}) = \boldsymbol{0}\}$ and defined by $\mathcal{S} = \{\boldsymbol{x} \in \mathbb{R}^n | \|\tilde{\boldsymbol{J}}_{\varphi x}(\boldsymbol{x})\| \leq \sqrt{2}\}$, *i.e.*, the region of the workspace where the angle, say α , between the normal to the surface $\boldsymbol{J}_{\varphi x}$ and its estimate $\hat{\boldsymbol{J}}_{\varphi x}$ is at most 90°, or equivalently $0 \leq \alpha \leq \pi/2$ (see Figure 15). Notice that this implies that the region \mathcal{S} must be taken into account in the definition of the region \mathcal{D} . As can be seen in Figure 16 (recalling that $\|\hat{\boldsymbol{J}}_{\varphi x}\| = \|\boldsymbol{J}_{\varphi x}\| = 1$), from the cosine rule one has (for $\alpha \leq \pi/2$)

$$\|\tilde{\boldsymbol{J}}_{\varphi \mathbf{x}}\|^{2} = 2\left(1 - \cos(\alpha)\right) \le 2\left(1 - \cos^{2}(\alpha)\right) = 2\|\hat{\boldsymbol{Q}}_{\mathbf{x}}\tilde{\boldsymbol{J}}_{\varphi \mathbf{x}}^{\mathrm{T}}\|^{2}, \qquad (98)$$



Figure 16: Projections

since from the same figure it can be seen that $\|\hat{\boldsymbol{Q}}_{x}\tilde{\boldsymbol{J}}_{\varphi x}^{T}\| = \cos(\pi/2 - \alpha) = \sin(\alpha)$. Hence (98) implies

$$\|\tilde{\boldsymbol{J}}_{\varphi \mathbf{x}}\| \le \sqrt{2} \|\hat{\boldsymbol{Q}}_{\mathbf{x}}\tilde{\boldsymbol{J}}_{\varphi \mathbf{x}}^{\mathrm{T}}\|.$$
(99)

Moreover, given that the robot always exerts force over the surface, there must exist a constant, say c_{λ} , such that $\lambda \geq c_{\lambda} > 0, \forall t \geq t_0$. Also, since it has been proven that $\hat{\lambda}$ is bounded in \mathcal{D} , after (23) there must exist a constant, say $c_{\hat{\lambda}}$, such that $0 \leq \hat{\lambda} \leq c_{\hat{\lambda}} < \infty, \forall t \geq t_0$. Therefore, (97) satisfies

$$\dot{V}_{\varphi \mathbf{x}} \leq -\frac{\gamma c_{\lambda}}{\hat{\lambda} + \epsilon} \| \hat{\boldsymbol{Q}}_{\mathbf{x}} \tilde{\boldsymbol{J}}_{\varphi \mathbf{x}}^{\mathrm{T}} \| \left(\| \hat{\boldsymbol{Q}}_{\mathbf{x}} \tilde{\boldsymbol{J}}_{\varphi \mathbf{x}}^{\mathrm{T}} \| - \frac{\lambda_{\mathrm{H}}}{c_{\lambda}} \| \boldsymbol{J}^{-\mathrm{T}}(\boldsymbol{q}_{1}) \| \| \tilde{\boldsymbol{z}}_{1} \| - \frac{c_{\hat{\lambda}} + \epsilon}{\gamma c_{\lambda}} \sqrt{2} v_{\mathbf{x}} \right)$$

$$\tag{100}$$

By choosing the eigenvalues of \boldsymbol{A} in (47) far away on the left in the complex plane, one guarantees that $(\tilde{\boldsymbol{q}}_1, \ldots, \tilde{\boldsymbol{q}}_1^{(p+2)})$ can be made arbitrarily small independently of the values of k_v and $k_{\rm Fi}$ (see (90)). Notice that this implies after (42) that $\boldsymbol{f}(t)$, and hence $\tilde{\boldsymbol{z}}_1$ in (41), can also be made arbitrarily small, which in turn implies that $(\lambda_{\rm H}/c_{\lambda}) \|\boldsymbol{J}^{-\rm T}(\boldsymbol{q}_1)\| \|\tilde{\boldsymbol{z}}_1\|$ can be made arbitrarily small as well since after Assumption 2.1 $\boldsymbol{J}^{-1}(\boldsymbol{q}_1)$ always exists. Finally, the term $(c_{\hat{\lambda}} + \epsilon/\gamma c_{\lambda})\sqrt{2}v_x$ can be made arbitrarily small by setting γ sufficiently large. By defining

$$c_{\mathbf{Q}} \triangleq \frac{\lambda_{\mathrm{H}}}{c_{\lambda}} \| \boldsymbol{J}^{-\mathrm{T}}(\boldsymbol{q}_{1}) \| \| \tilde{\boldsymbol{z}}_{1} \| + \frac{c_{\lambda} + \epsilon}{\gamma c_{\lambda}} \sqrt{2} v_{\mathbf{x}}, \qquad (101)$$

one has

$$\|\hat{\boldsymbol{Q}}_{x}\tilde{\boldsymbol{J}}_{\varphi x}^{\mathrm{T}}\| \ge c_{\mathrm{Q}} \implies \dot{V}_{\varphi x} \le 0.$$
 (102)

After (94), it can be stated

$$\frac{1}{2} \|\tilde{\boldsymbol{J}}_{\varphi \mathbf{x}}\|^2 \le V_{\varphi \mathbf{x}} \le \frac{1}{2} \|\tilde{\boldsymbol{J}}_{\varphi \mathbf{x}}\|^2, \qquad (103)$$

which means that for $\dot{V}_{\varphi \mathbf{x}} \leq 0$

$$\|\tilde{\boldsymbol{J}}_{\varphi\mathbf{x}}(t)\| \le \|\tilde{\boldsymbol{J}}_{\varphi\mathbf{x}}(t_0)\|, \quad \forall t \ge t_0.$$
(104)

In the case $\|\hat{Q}\tilde{J}_{\varphi \mathbf{x}}(t)\| = \sqrt{2}c_{\mathbf{Q}}$, after (99), the ultimate bound for $\|\tilde{J}_{\varphi \mathbf{x}}\|$ is given by

$$\|\tilde{\boldsymbol{J}}_{\varphi \mathbf{x}}\| \le \sqrt{2}c_{\mathbf{Q}} \,. \tag{105}$$

Furthermore, the smaller $\|\hat{\boldsymbol{Q}}_{\mathbf{x}}\tilde{\boldsymbol{J}}_{\varphi\mathbf{x}}^{\mathrm{T}}\|$, the smaller $\|\tilde{\boldsymbol{J}}_{\varphi\mathbf{x}}\|$. By adding the functions defined in (86), (92) and (94) we obtain the positive definite function

$$V = V_{a} + V_{s} + V_{\varphi x} = \boldsymbol{x}_{o}^{T} \boldsymbol{P}_{o} \boldsymbol{x}_{o} + \frac{1}{2} \boldsymbol{s}^{T} \boldsymbol{H}(\boldsymbol{q}) \boldsymbol{s} + \frac{1}{4} \frac{k_{Fi}}{k_{v}} (\Delta \bar{F})^{2} + \frac{1}{2} \tilde{\boldsymbol{J}}_{\varphi x} \tilde{\boldsymbol{J}}_{\varphi x}^{T}$$
$$= \boldsymbol{y}^{T} \begin{bmatrix} \boldsymbol{P}_{o} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \frac{1}{2} \boldsymbol{H}(\boldsymbol{q}) & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \frac{1}{4} \frac{k_{Fi}}{k_{v}} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & 0 & \frac{1}{2} \boldsymbol{I} \end{bmatrix} \boldsymbol{y} = \boldsymbol{y}^{T} \boldsymbol{M}(\boldsymbol{q}) \boldsymbol{y} , \qquad (106)$$

where each O is a matrix or vector of zeros with appropriate dimensions. Given Property 2.1, we can find two positive constants, $\lambda_{\rm m}$ and $\lambda_{\rm M}$, such that

$$\lambda_{\mathrm{m}} \|\boldsymbol{y}\|^{2} \leq V(\boldsymbol{y}) \leq \lambda_{\mathrm{M}} \|\boldsymbol{y}\|^{2} \,. \tag{107}$$

After (88), (93) and (100), the time derivative of (106) along the trajectories of the system fulfils

$$\dot{V} \leq -\lambda_{\min}(\boldsymbol{Q}_{o}) \|\boldsymbol{x}_{o}\| \left(\|\boldsymbol{x}_{o}\| - \frac{2\lambda_{\max}(\boldsymbol{P}_{o})\|\boldsymbol{B}\|r_{\max}}{\lambda_{\min}(\boldsymbol{Q}_{o})} \right) - k_{v} \|\boldsymbol{s}\| \left(\|\boldsymbol{s}\| - \frac{c_{a}}{k_{v}} \right) - \frac{1}{4} \frac{k_{Fi}^{2} c_{\varphi}^{-}}{k_{v}} |\Delta \bar{F}| \left(|\Delta \bar{F}| - \frac{2\lambda_{H} c_{\varphi}^{+}}{k_{Fi} c_{\varphi}^{-}} \|\tilde{\boldsymbol{z}}_{1}\| \right) - \frac{\gamma c_{\lambda}}{\hat{\lambda} + \epsilon} \|\hat{\boldsymbol{Q}}_{x} \tilde{\boldsymbol{J}}_{\varphi x}^{T}\| \left(\|\hat{\boldsymbol{Q}}_{x} \tilde{\boldsymbol{J}}_{\varphi x}^{T}\| - \frac{\lambda_{H}}{c_{\lambda}} \|\boldsymbol{J}^{-T}(\boldsymbol{q}_{1})\| \|\tilde{\boldsymbol{z}}_{1}\| - \frac{c_{\lambda} + \epsilon}{\gamma c_{\lambda}} \sqrt{2} v_{x} \right).$$
(108)

According to the previous discussion, the terms

$$\frac{2\lambda_{\max}(\boldsymbol{P}_{o})\|\boldsymbol{B}\|r_{\max}}{\lambda_{\min}(\boldsymbol{Q}_{o})}, \ \frac{c_{a}}{k_{v}}, \ \frac{2\lambda_{H}c_{\varphi}^{+}}{k_{Fi}c_{\varphi}^{-}}\|\tilde{\boldsymbol{z}}_{1}\|, \ \frac{\lambda_{H}}{c_{\lambda}}\|\boldsymbol{J}^{-T}(\boldsymbol{q}_{1})\|\|\tilde{\boldsymbol{z}}_{1}\|, \ \text{and} \ \frac{c_{\lambda}+\epsilon}{\gamma c_{\lambda}}\sqrt{2}v_{x}$$

can be made arbitrarily small in \mathcal{D} by choosing the eigenvalues of \mathbf{A} in (47) far away on the left in the complex plane and the gains $k_{\rm v}$, $k_{\rm Fi}$, and γ large

enough, as well as by choosing y_{\max} small enough. Overall, we can always find a positive arbitrarily small constant μ such that

$$\dot{V} \le 0 \quad \text{if} \quad \|\boldsymbol{y}\| \ge \mu \,.$$
 (109)

Once $\|\boldsymbol{y}\| = \mu$, from (107) the maximum value that $\|\boldsymbol{y}\|$ can take is given by

$$\lambda_{\mathrm{m}} \| \boldsymbol{y} \|^{2} \leq V(\boldsymbol{y}) \leq \lambda_{\mathrm{M}} \mu^{2} \implies \| \boldsymbol{y} \| \leq \sqrt{\frac{\lambda_{\mathrm{M}}}{\lambda_{\mathrm{m}}}} \mu \triangleq b, \quad (110)$$

where b is the ultimate bound of the state y. Recall that it must be guaranteed that $\|y\| \leq y_{\max}, \forall t \geq t_0$. This can be done by setting gains large enough to satisfy

$$\mu < \sqrt{\frac{\lambda_{\rm m}}{\lambda_{\rm M}}} y_{\rm max} \,. \tag{111}$$

Also, the initial condition must satisfy

$$\|\boldsymbol{y}(t_0)\| < \sqrt{\frac{\lambda_{\mathrm{m}}}{\lambda_{\mathrm{M}}}} y_{\mathrm{max}}$$
(112)

to guarantee that \boldsymbol{y} never leaves the region \mathcal{D} .

d) Finally, since b can be made arbitrarily small, then $\|\boldsymbol{y}\|$ can be made arbitrarily close to zero. This implies the that observation errors $\tilde{\boldsymbol{q}}_1, \ldots, \tilde{\boldsymbol{q}}_1^{(p+2)}$ are made approximately zero. Therefore, after (35), $\tilde{\boldsymbol{q}}_2 \approx \mathbf{0} \implies \boldsymbol{q}_2 \approx \hat{\boldsymbol{q}}_2$, *i.e.*, $\hat{\boldsymbol{q}}_2$ is an arbitrarily close estimation of the vector of joint velocities \boldsymbol{q}_2 . Also, we have proved that $\tilde{\boldsymbol{z}}_1 \approx \mathbf{0}$, which after (10) and (23) means that $\lambda - \hat{\lambda} \approx 0 \implies \hat{\lambda} \approx \lambda$ for $\lambda > 0$, which implies arbitrary close estimation of the contact force. Since the ultimate bound of $\|\boldsymbol{y}\|$, and therefore of $\|\boldsymbol{s}\|$ and $\|\Delta \bar{F}\|$, can be made arbitrarily small, from (56) one can see that the ultimate bound of $\Delta \lambda$ must be arbitrarily small as well, implying that force tracking is achieved.