A master–slave teleoperation system, in which the slave manipulator is interacting with a rigid surface with unknown geometry, is considered. It is assumed that neither force nor velocity measurements at the slave side are available. To deal with this problem, an extended–state high–gain observer is proposed to estimate in an arbitrary close manner the velocity and force signals. At the same time, the gradient vector for the remote surface is on line estimated and utilised into an hybrid position/force controller based on the orthogonal decomposition of the task space. A formal proof is presented, which guarantees ultimate boundedness of the state of the system, with arbitrarily small ultimate bound. Furthermore, it is established the transparency of the teleoperation system that, roughly speaking, gives the human operator the sensation of being interacting directly with the remote surface. The proposed scheme is validated through numerical simulations.

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Transparent bilateral teleoperation interacting with unknown remote surfaces with a force/velocity observer design

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1. Introduction

In the last decades there have been a lot of research regarding robotic teleoperation systems. Starting with the seminal work of Goertz (Goertz and Bevilacqua, 2002), it has been a very prolific area since then (see Hokayem and Spong (2006) for a historical survey up to 2006). The majority of the published works are dedicated to the problem of delays in the communication channel, for which the main goal is to guarantee stability in presence of both constant and variable delays (Anderson and Spong, 1989; Niemeyer and Slotine, 1991; Nuño et al., 2011, 2009).

Another important goal when designing a controller for a teleoperator is *transparency*. A robotic teleoperation system is said to be *transparent* if the dynamics of the manipulators are not felt by the human operator, giving him/her a sensation of *telepresence*, *i.e.,* the impression of being directly manipulating the remote object/environment. In this aspect, some conditions for obtaining the ideal transparency have been established in Lawrence (1993); Yokokohji and Yoshikawa (1994). Remarkably, in Lawrence (1993) it is stated that robust stabilisation (passivity) and transparency are two contradictory goals even in the non–delayed scenario.

When the slave manipulator is in contact with a rigid surface it is commonly assumed that the geometric description of this surface is exactly known. In fact, an error in this description could easily lead to instability (Wang and McClamroch, 1994). To deal with the uncertainty in the model of the remote surface, some solutions based on adaptive control have been proposed (Lee and Chung, 1998; Liu et al., 2014, 2010).

In a different direction, many works have focused on the elimination of sensors for a variety of reasons (to reduce costs, weight, size). In this context, there have been some efforts dedicated to removing the necessity of a force sensor. For example, in Hashtrudi-Zaad and Salcudean (1996) it is presented a controller that does not need force, but requires velocity and acceleration measurements. For non–stiff environments, a controller that does not require velocity nor force measurements is presented in Daly and Wang (2014) for a teleoperation system with time delays.

In this work, a master–slave teleoperation system when the geometry of the remote surface

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is unknown and when neither force nor velocity measurements at the slave side are available is considered. The goal pursued here is the transparency of the system, for which it will be followed the *virtual surfaces* approach introduced in Rodríguez-Angeles et al. (2015). The extended–state high–gain observer reported in Gutiérrez-Giles and Arteaga-Pérez (2014) is employed to estimate the joint velocities and contact force at the slave side. Besides, it is introduced a local estimator to deal with the problem of the remote surface unknown geometry.

The paper is organised as follows: in Section 2 a mathematical model for the teleoperation system is introduced as well as some of its properties. Section 3 presents the main result of this work, which accounts for the observer, controller and surface estimator design for which ultimate boundedness of all signals of interest with arbitrary small ultimate bound is formally proven. This allows to guarantee arbitrarily close tracking of position and force, along with an on line estimation of contact force, slave manipulator joint velocities and the gradient vector of the remote surface. A numerical simulation to validate the proposed approach is presented in Section 4. Finally, some concluding remarks and guidelines for future work are given in Section 6.

2. Mathematical model and properties

Consider a master–slave teleoperation system described by

$$\boldsymbol{H}_{\mathrm{m}}(\boldsymbol{q}_{\mathrm{m}})\ddot{\boldsymbol{q}}_{\mathrm{m}} + \boldsymbol{C}_{\mathrm{m}}(\boldsymbol{q}_{\mathrm{m}}, \dot{\boldsymbol{q}}_{\mathrm{m}})\dot{\boldsymbol{q}}_{\mathrm{m}} + \boldsymbol{D}_{\mathrm{m}}\dot{\boldsymbol{q}}_{\mathrm{m}} + \boldsymbol{g}_{\mathrm{m}}(\boldsymbol{q}_{\mathrm{m}}) = \boldsymbol{\tau}_{\mathrm{m}} - \boldsymbol{\tau}_{\mathrm{h}} \tag{1}$$

$$\boldsymbol{H}_{\mathrm{s}}(\boldsymbol{q}_{\mathrm{s}})\ddot{\boldsymbol{q}}_{\mathrm{s}} + \boldsymbol{C}_{\mathrm{s}}(\boldsymbol{q}_{\mathrm{s}}, \dot{\boldsymbol{q}}_{\mathrm{s}})\dot{\boldsymbol{q}}_{\mathrm{s}} + \boldsymbol{D}_{\mathrm{s}}\dot{\boldsymbol{q}}_{\mathrm{s}} + \boldsymbol{g}_{\mathrm{s}}(\boldsymbol{q}_{\mathrm{s}}) = \boldsymbol{\tau}_{\mathrm{s}} + \boldsymbol{J}_{\varphi\mathrm{s}}^{\mathrm{T}}(\boldsymbol{q}_{\mathrm{s}})\boldsymbol{\lambda}_{\mathrm{s}}\,, \tag{2}$$

where the sub–indexes m and s denote the master and slave manipulators, respectively. For $i = \mathrm{m}, \mathrm{s}$, $\boldsymbol{q}_i \in \mathbb{R}^n$ is the vector of generalised coordinates, $\boldsymbol{H}_i(\boldsymbol{q}_i) \in \mathbb{R}^{n\times n}$ is the inertia matrix, $\boldsymbol{C}_i(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i)\dot{\boldsymbol{q}}_i \in \mathbb{R}^n$ is the vector of Coriolis and centripetal forces, $\boldsymbol{D}_i \in \mathbb{R}^{n\times n}$ is a diagonal matrix of viscous friction coefficients, $\boldsymbol{g}_i(\boldsymbol{q}_i) \in \mathbb{R}^n$ is the vector of gravitational torques, $\boldsymbol{\tau}_i \in \mathbb{R}^n$ is the vector of generalised inputs, $\boldsymbol{\tau}_{\mathrm{h}} \in \mathbb{R}^n$ is the torque applied by the human operator over the master robot, $\boldsymbol{\lambda}_{\mathrm{s}} \in \mathbb{R}^m$ is a vector of Lagrange multipliers (physically represents the contact force over the rigid surface), and $\boldsymbol{J}_{\varphi\mathrm{s}}(\boldsymbol{q}_{\mathrm{s}}) = \nabla\boldsymbol{\varphi}_{\mathrm{s}}(\boldsymbol{q}_{\mathrm{s}}) \in \mathbb{R}^{m\times n}$ is the gradient of the remote surface described by the constraint

$$\boldsymbol{\varphi}_{\mathrm{s}}(\boldsymbol{q}_{\mathrm{s}}) = \boldsymbol{0}\,, \tag{3}$$

that can also be expressed in the slave manipulator Cartesian coordinates as

$$\boldsymbol{\varphi}_{\mathrm{s}}(\boldsymbol{x}_{\mathrm{s}}) = \boldsymbol{0}\,, \tag{4}$$

where $\boldsymbol{x}_{\mathrm{s}} \in \mathbb{R}^n$ is the vector of end–effector coordinates. It is assumed that a suitable normalisation is done for the gradient of this constraint, $\boldsymbol{J}_{\varphi\mathrm{xs}}(\boldsymbol{x}_{\mathrm{s}}) = \nabla\boldsymbol{\varphi}_{\mathrm{s}}(\boldsymbol{x}_{\mathrm{s}}) \in \mathbb{R}^{m\times n}$, to be unitary. These two gradients are related by

$$\boldsymbol{J}_{\varphi\mathrm{s}}(\boldsymbol{q}_{\mathrm{s}}) = \boldsymbol{J}_{\varphi\mathrm{xs}}(\boldsymbol{x}_{\mathrm{s}})\boldsymbol{J}_{\mathrm{s}}(\boldsymbol{q}_{\mathrm{s}})\,, \tag{5}$$

where $\boldsymbol{J}_{\mathrm{s}}(\boldsymbol{q}_{\mathrm{s}})$ is the analytic Jacobian of the manipulator.

It will be employed an orthogonal decomposition of the task space, carried out as follows. Let $\boldsymbol{J}_{\varphi\mathrm{s}}^{+} \triangleq \boldsymbol{J}_{\varphi\mathrm{s}}^{\mathrm{T}}\left(\boldsymbol{J}_{\varphi\mathrm{s}}\boldsymbol{J}_{\varphi\mathrm{s}}^{\mathrm{T}}\right)^{-1}$, $\boldsymbol{P}_{\mathrm{s}}(\boldsymbol{q}_{\mathrm{s}}) = \boldsymbol{J}_{\varphi\mathrm{s}}^{+}\boldsymbol{J}_{\varphi\mathrm{s}}$, and $\boldsymbol{Q}_{\mathrm{s}}(\boldsymbol{q}_{\mathrm{s}}) = \boldsymbol{I}_{n\times n} - \boldsymbol{P}_{\mathrm{s}}(\boldsymbol{q}_{\mathrm{s}})$, whence $\boldsymbol{P}_{\mathrm{s}}$ and $\boldsymbol{Q}_{\mathrm{s}}$ are projection matrices, *i.e.,* $\boldsymbol{Q}_{\mathrm{s}}\boldsymbol{P}_{\mathrm{s}} = \boldsymbol{O}$, $\boldsymbol{Q}_{\mathrm{s}}\boldsymbol{J}_{\varphi\mathrm{s}} = \boldsymbol{O}$, and $\boldsymbol{J}_{\varphi\mathrm{s}}\boldsymbol{Q}_{\mathrm{s}} = \boldsymbol{O}$. Moreover, $\boldsymbol{Q}_{\mathrm{s}}\boldsymbol{Q}_{\mathrm{s}} = \boldsymbol{Q}_{\mathrm{s}}$ and $\boldsymbol{P}_{\mathrm{s}}\boldsymbol{P}_{\mathrm{s}} = \boldsymbol{P}_{\mathrm{s}}$. Then, the following property can be stated (Martínez-Rosas et al., 2006).

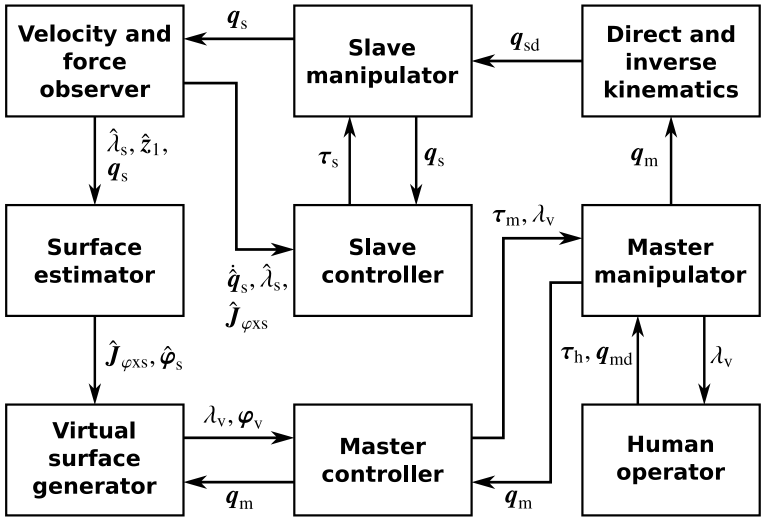

Figure 1. Block diagram of the proposed scheme.

Property 1: The joint velocity vector $\dot{\boldsymbol{q}}_\mathrm{s}$ satisfies

$$\dot{\boldsymbol{q}}_\mathrm{s} = \boldsymbol{Q}_\mathrm{s}(\boldsymbol{q}_\mathrm{s})\dot{\boldsymbol{q}}_\mathrm{s} + \boldsymbol{P}_\mathrm{s}(\boldsymbol{q}_\mathrm{s})\dot{\boldsymbol{q}}_\mathrm{s} = \boldsymbol{Q}_\mathrm{s}(\boldsymbol{q}_\mathrm{s})\dot{\boldsymbol{q}}_\mathrm{s}\,. \tag{6}$$

□

Now, it is introduced an assumption that must be taken into account at the trajectory planning stage.

Assumption 1: *None of the manipulators reach any singularity, so that* $\boldsymbol{J}_i(\boldsymbol{q}_i)$, $i = \mathrm{m,s}$ *are always invertible* . □

For simplicity's sake, consider that the manipulators have only revolute joints. Thereupon, each one satisfies the following standard properties (Arteaga-Pérez, 1998).

Property 2: The inertia matrix $\boldsymbol{H}_i(\boldsymbol{q}_i)$ is symmetric positive definite and $\forall \boldsymbol{q}_i, \boldsymbol{y} \in \mathbb{R}^n$ it holds $\lambda_\mathrm{h}\|\boldsymbol{y}\|^2 \leq \boldsymbol{y}^\mathrm{T}\boldsymbol{H}_i(\boldsymbol{q}_i)\boldsymbol{y} \leq \lambda_\mathrm{H}\|\boldsymbol{y}\|^2$, with $0 < \lambda_\mathrm{h} \leq \lambda_\mathrm{H} < \infty$. □

Property 3: With a proper definition of $\boldsymbol{C}_i(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i)$, the matrix $\dot{\boldsymbol{H}}_i(\boldsymbol{q}_i) - 2\boldsymbol{C}_i(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i)$ is skew symmetric. □

Property 4: The vector $\boldsymbol{C}_i(\boldsymbol{q}_i, \boldsymbol{x}_i)\boldsymbol{y}_i$ satisfies $\boldsymbol{C}_i(\boldsymbol{q}_i, \boldsymbol{x})\boldsymbol{y} = \boldsymbol{C}_i(\boldsymbol{q}_i, \boldsymbol{y})\boldsymbol{x}\,, \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. □

3. Main result

In this work, it is considered the case of one–dimensional constraints ($\boldsymbol{\varphi}_\mathrm{s} : \mathbb{R}^n \to \mathbb{R}$), so that $\boldsymbol{\lambda}_\mathrm{s} = \lambda_\mathrm{s} \in \mathbb{R}$ represents the contact force of the slave manipulator over the remote surface. To achieve trajectory tracking of both position and force and *transparency* of the teleoperation system (Lawrence, 1993), it will be followed the *virtual surfaces* approach (Rodríguez-Angeles et al., 2015). A diagram of the proposed scheme is shown in Figure 1.

3.1 *Control and Observer design*

First, consider a state–space representation of the slave manipulator model (2)

$$\dot{\boldsymbol{q}}_{\rm s} = \boldsymbol{q}_{2\rm s} \tag{7}$$
$$\dot{\boldsymbol{q}}_{2\rm s} = \boldsymbol{H}_{\rm s}^{-1}(\boldsymbol{q}_{\rm s})\left(\boldsymbol{\tau}_{\rm s} - \boldsymbol{N}_{\rm s}(\boldsymbol{q}_{\rm s}, \boldsymbol{q}_{2\rm s})\right) + \boldsymbol{z}_1\,, \tag{8}$$

where $\boldsymbol{N}_{\rm s}(\boldsymbol{q}_{\rm s}, \boldsymbol{q}_{2\rm s}) = \boldsymbol{C}_{\rm s}(\boldsymbol{q}_{\rm s}, \boldsymbol{q}_{2\rm s})\boldsymbol{q}_{2\rm s} + \boldsymbol{D}_{\rm s}\boldsymbol{q}_{2\rm s} + \boldsymbol{g}_{\rm s}(\boldsymbol{q}_{\rm s})$ and

$$\boldsymbol{z}_1 = \boldsymbol{H}_{\rm s}^{-1}(\boldsymbol{q}_{\rm s})\boldsymbol{J}_{\varphi\rm s}^{\rm T}(\boldsymbol{q}_{\rm s})\lambda_{\rm s}\,. \tag{9}$$

Assume that $\boldsymbol{z}_1$ can be described by the following dynamic internal model (see Gutiérrez-Giles and Arteaga-Pérez, 2014, Section 3.1, for details)

$$\dot{\boldsymbol{z}}_1 = \boldsymbol{z}_2 \tag{10}$$
$$\vdots$$
$$\dot{\boldsymbol{z}}_{p-1} = \boldsymbol{z}_p \tag{11}$$
$$\dot{\boldsymbol{z}}_p = \boldsymbol{r}^{(p)}(t)\,. \tag{12}$$

Then, the following extended–state high–gain observer can be employed to reconstruct the velocity and force in the remote side of the teleoperation system (Gutiérrez-Giles and Arteaga-Pérez, 2014)

$$\dot{\hat{\boldsymbol{q}}}_{\rm s} = \hat{\boldsymbol{q}}_{2\rm s} + \boldsymbol{\lambda}_{p+1}\tilde{\boldsymbol{q}}_{\rm s} \tag{13}$$
$$\dot{\hat{\boldsymbol{q}}}_{2\rm s} = \boldsymbol{H}_{\rm s}^{-1}(\boldsymbol{q}_{\rm s})\left(\boldsymbol{\tau}_{\rm s} - \boldsymbol{N}_{\rm s}(\boldsymbol{q}_{\rm s}, \hat{\boldsymbol{q}}_{2\rm s})\right) + \hat{\boldsymbol{z}}_1 + \boldsymbol{\lambda}_p\tilde{\boldsymbol{q}}_{\rm s} \tag{14}$$
$$\dot{\hat{\boldsymbol{z}}}_1 = \hat{\boldsymbol{z}}_2 + \boldsymbol{\lambda}_{p-1}\tilde{\boldsymbol{q}}_{\rm s} \tag{15}$$
$$\vdots$$
$$\dot{\hat{\boldsymbol{z}}}_{p-1} = \hat{\boldsymbol{z}}_p + \boldsymbol{\lambda}_1\tilde{\boldsymbol{q}}_{\rm s} \tag{16}$$
$$\dot{\hat{\boldsymbol{z}}}_p = \boldsymbol{\lambda}_0\tilde{\boldsymbol{q}}_{\rm s}\,, \tag{17}$$

where $\tilde{\boldsymbol{q}}_{\rm s} \triangleq \boldsymbol{q}_{\rm s} - \hat{\boldsymbol{q}}_{\rm s}$ and $\boldsymbol{N}_{\rm s}(\boldsymbol{q}_{\rm s}, \hat{\boldsymbol{q}}_{2\rm s}) \triangleq \boldsymbol{C}_{\rm s}(\boldsymbol{q}_{\rm s}, \hat{\boldsymbol{q}}_{2\rm s})\hat{\boldsymbol{q}}_{2\rm s} + \boldsymbol{D}_{\rm s}\hat{\boldsymbol{q}}_{2\rm s} + \boldsymbol{g}_{\rm s}(\boldsymbol{q}_{\rm s})$. An estimation of the joint velocity is obtained directly by $\hat{\boldsymbol{q}}_{2\rm s}$. The contact force can be approximated by

$$\hat{\lambda}_{\rm s} = \|\boldsymbol{J}_{\rm s}^{-\rm T}(\boldsymbol{q}_{\rm s})\boldsymbol{H}_{\rm s}(\boldsymbol{q}_{\rm s})\hat{\boldsymbol{z}}_1\|\,. \tag{18}$$

Recall that it is supposed that there is no available information about the geometry of the surface. Nevertheless, the following assumption is made in order to design an estimator and to carry out the corresponding stability analysis.

Assumption 2: *The remote surface described by (4) is smooth.* □

Accordingly, an on line estimator of the remote surface gradient in end–effector coordinates is proposed as

$$\dot{\hat{\boldsymbol{J}}}_{\varphi\rm xs}^{\rm T} = \left(\frac{\gamma}{\hat{\lambda}_{\rm s} + \epsilon}\right)\hat{\boldsymbol{Q}}_{\rm xs}\boldsymbol{J}_{\rm s}^{-\rm T}(\boldsymbol{q}_{\rm s})\boldsymbol{H}_{\rm s}(\boldsymbol{q}_{\rm s})\hat{\boldsymbol{z}}_1\,, \tag{19}$$

where $\gamma > 0$ is the estimation gain, $\epsilon > 0$ is a (small) positive constant to avoid division by zero, and $\hat{\boldsymbol{Q}}_{\rm xs} \triangleq \boldsymbol{I}_{n\times n} - \hat{\boldsymbol{P}}_{\rm xs}$, with $\hat{\boldsymbol{P}}_{\rm xs} \triangleq \hat{\boldsymbol{J}}^{+}_{\varphi{\rm xs}}\hat{\boldsymbol{J}}_{\varphi{\rm xs}}$, and $\hat{\boldsymbol{J}}^{+}_{\varphi{\rm xs}} = \hat{\boldsymbol{J}}^{\rm T}_{\varphi{\rm xs}}(\hat{\boldsymbol{J}}_{\varphi{\rm xs}}\hat{\boldsymbol{J}}^{\rm T}_{\varphi{\rm xs}})^{-1}$. It is claimed that $\|\hat{\boldsymbol{J}}_{\varphi{\rm xs}}(t)\| = \|\hat{\boldsymbol{J}}_{\varphi{\rm xs}}(t_0))\|, \forall t \geq t_0$. To see this, compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{\boldsymbol{J}}_{\varphi{\rm xs}}\|^2 = 2\hat{\boldsymbol{J}}_{\varphi{\rm xs}}\dot{\hat{\boldsymbol{J}}}^{\rm T}_{\varphi{\rm xs}} = 2\left(\frac{\gamma}{\hat{\lambda}_{\rm s}+\epsilon}\right)\hat{\boldsymbol{J}}_{\varphi{\rm xs}}\hat{\boldsymbol{Q}}_{\rm xs}\boldsymbol{J}_{\rm s}^{-\rm T}(\boldsymbol{q}_{\rm s})\boldsymbol{H}_{\rm s}(\boldsymbol{q}_{\rm s})\hat{\boldsymbol{z}}_1 = 0\,, \tag{20}$$

since $\hat{\boldsymbol{J}}_{\varphi{\rm xs}}\hat{\boldsymbol{Q}}_{\rm xs} = 0$. Therefore it is appropriate to set the initial condition of the estimator to satisfy $\|\hat{\boldsymbol{J}}_{\varphi{\rm xs}}(t_0))\| = 1$. Notice that (20) is true regardless how accurate the surface reconstruction can be. The above matrices can be expressed in joint coordinates by defining

$$\hat{\boldsymbol{J}}_{\varphi{\rm s}} \triangleq \hat{\boldsymbol{J}}_{\varphi{\rm xs}}\boldsymbol{J}_{\rm s}\,, \tag{21}$$

with $\hat{\boldsymbol{P}}_{\rm s}$, $\hat{\boldsymbol{Q}}_{\rm s}$, and $\hat{\boldsymbol{J}}^{+}_{\varphi{\rm s}}$ defined analogously to $\hat{\boldsymbol{P}}_{\rm xs}$, $\hat{\boldsymbol{Q}}_{\rm xs}$, and $\hat{\boldsymbol{J}}^{+}_{\varphi{\rm xs}}$.

Assume that a desired force $\lambda_{\rm sd} > 0$ is commanded by the human operator. Define the position tracking error in joint coordinates at the slave side as

$$\boldsymbol{e}_{\rm s} \triangleq \boldsymbol{q}_{\rm s} - \boldsymbol{q}_{\rm sd}\,, \tag{22}$$

where $\boldsymbol{q}_{\rm sd}$ is obtained by solving the inverse kinematics of the slave robot with the pose of the master manipulator as the desired pose. Taking this into account, the following control law is proposed for the slave manipulator

$$\boldsymbol{\tau}_{\rm s} = -\boldsymbol{K}_{\rm ps}\boldsymbol{e}_{\rm s} - \boldsymbol{K}_{\rm vs}(\hat{\boldsymbol{q}}_{\rm s2} - \dot{\boldsymbol{q}}_{\rm sd}) - \hat{\boldsymbol{Q}}_{\rm s}\boldsymbol{K}_{\rm is}\int_{t_0}^{t}\boldsymbol{e}_{\rm s}\,\mathrm{d}\vartheta - \hat{\boldsymbol{J}}^{\rm T}_{\varphi s}\lambda_{\rm sd} + \hat{\boldsymbol{J}}^{+}_{\varphi s}k_{\rm Fis}\Delta\bar{F}_{\rm s}\,, \tag{23}$$

where $\boldsymbol{K}_{\rm ps}, \boldsymbol{K}_{\rm vs}, \boldsymbol{K}_{\rm is} \in \mathbb{R}^{n\times n}$ are diagonal positive definite matrices of constant gains, $k_{\rm Fis} > 0$ is the integral force control gain and

$$\Delta\bar{\lambda}_{\rm s} \triangleq \hat{\lambda}_{\rm s} - \lambda_{\rm sd} \tag{24}$$

$$\Delta\bar{F}_{\rm s} \triangleq \int_{t_0}^{t}\Delta\bar{\lambda}_{\rm s}\,\mathrm{d}\vartheta\,. \tag{25}$$

Since the geometry of the remote environment is unknown a *virtual surface* cannot be created directly as in Rodríguez-Angeles et al. (2015), but the information of the estimator (19) must be employed. In this work, a local approximation of constraint (4) is considered as described in Pliego-Jiménez and Arteaga-Pérez (2015). It is assumed that the human operator is responsible for the desired trajectory by moving the master manipulator. Then, the desired trajectory in task space coordinates for the slave manipulator is given by $\boldsymbol{x}_{\rm sd} = \boldsymbol{x}_{\rm m}$ (with a possible scale factor), and the approximation of the virtual surface constraint is thus proposed as

$$\hat{\varphi}_{\rm v} = \hat{\boldsymbol{J}}_{\varphi{\rm xs}}(\boldsymbol{x}_{\rm sd} - \boldsymbol{x}_{\rm sa})\,, \tag{26}$$

where $\boldsymbol{x}_{\rm sa}$ is the output of the first order filter

$$\dot{\boldsymbol{x}}_{\rm sa} = -\eta\boldsymbol{x}_{\rm sa} + \eta\boldsymbol{x}_{\rm s}\,, \tag{27}$$

with $\eta > 0$ (see Pliego-Jiménez and Arteaga-Pérez, 2015, for details). The first and second derivatives of the virtual constraint can be approximated by

$$\hat{\dot{\varphi}}_{\rm v} = \hat{\boldsymbol{J}}_{\varphi {\rm xs}} \hat{\boldsymbol{q}}_{\rm s2} \tag{28}$$

$$\hat{\ddot{\varphi}}_{\rm v} = \dot{\hat{\boldsymbol{J}}}_{\varphi {\rm xs}} \hat{\boldsymbol{q}}_{\rm s2} + \hat{\boldsymbol{J}}_{\varphi {\rm xs}} \dot{\hat{\boldsymbol{q}}}_{\rm s2} \,. \tag{29}$$

To reflect the contact force to the operator, a Lagrange multiplier is computed iteratively as in Bayo and Avello (1994), *i.e.*,

$$\lambda_{{\rm v}_{(i+1)}} = \lambda_{{\rm v}_i} + \alpha_{\rm v} \left(\hat{\ddot{\varphi}}_{\rm v} + 2\xi\omega_{\rm n} \hat{\dot{\varphi}}_{\rm v} + \omega_{\rm n}^2 \hat{\varphi}_{\rm v} \right)_{(i+1)} \,, \quad i = 0, 1, 2, \dots \,, \tag{30}$$

where $\xi, \omega > 0$ and $\lambda_{{\rm v}_0} = 0$. To achieve the desired transparency of the teleoperation system, a dynamic cancellation of the dynamics of the master manipulator is carried out. On the other hand, the contact force must be reflected to the human operator through the master manipulator. Accordingly, the control law for the master robot is proposed as

$$\boldsymbol{\tau}_{\rm m} = \boldsymbol{H}(\boldsymbol{q}_{\rm m}) \ddot{\boldsymbol{q}}_{\rm m} + \boldsymbol{C}_{\rm m}(\boldsymbol{q}_{\rm m}, \dot{\boldsymbol{q}}_{\rm m}) \dot{\boldsymbol{q}}_{\rm m} + \boldsymbol{D}_{\rm m} \dot{\boldsymbol{q}}_{\rm m} + \boldsymbol{g}_{\rm m}(\boldsymbol{q}_{\rm m}) - \hat{\boldsymbol{J}}_{\varphi {\rm v}}^{\rm T} \left(k_{\rm Fv} \Delta\lambda_{\rm vs} + k_{\rm Fiv} \Delta F_{\rm vs} \right) \,, \tag{31}$$

where $k_{\rm Fv}, k_{\rm Fiv} > 0$, $\hat{\boldsymbol{J}}_{\varphi {\rm v}} \triangleq \hat{\boldsymbol{J}}_{\varphi {\rm s}}$ (with a possible scale factor), and

$$\Delta\lambda_{\rm vs} \triangleq \lambda_{\rm v} - \hat{\lambda}_{\rm s} \tag{32}$$

$$\Delta F_{\rm vs} \triangleq \int_{t_0}^{t} \Delta\lambda_{\rm vs} \, {\rm d}\vartheta \,. \tag{33}$$

Now, an assumption related with the behaviour of the human operator is introduced both to carry out the stability analysis of the teleoperation system and for simulation. Let $\boldsymbol{q}_{\rm d}(t)$ be the desired trajectory for the master manipulator in joint coordinates as wished by the human and $\boldsymbol{e}_{\rm m} \triangleq \boldsymbol{q}_{\rm m} - \boldsymbol{q}_{\rm d}$ the corresponding tracking error.

Assumption 3: *The human operator imposes the torque $\boldsymbol{\tau}_{\rm h}$ over the master manipulator according with the control law*

$$\boldsymbol{\tau}_{\rm h} = \hat{\boldsymbol{Q}}_{\rm v} \left\{ \boldsymbol{K}_{\rm ph} \boldsymbol{e}_{\rm m} + \boldsymbol{K}_{\rm vh} \dot{\boldsymbol{e}}_{\rm m} + \boldsymbol{K}_{\rm ih} \int_{t_0}^{t} \boldsymbol{e}_{\rm m} \, {\rm d}\vartheta \right\} + \hat{\boldsymbol{J}}_{\varphi {\rm v}}^{\rm T} \left\{ k_{\rm Fh} \Delta\lambda_{\rm vd} + k_{\rm Fih} \Delta F_{\rm vd} \right\} \,, \tag{34}$$

where $\hat{\boldsymbol{Q}}_{\rm v} \triangleq \boldsymbol{I}_{n\times n} - \hat{\boldsymbol{J}}_{\varphi {\rm v}}^{+} \hat{\boldsymbol{J}}_{\varphi {\rm v}}$, *with* $\hat{\boldsymbol{J}}_{\varphi {\rm v}}^{+} = \hat{\boldsymbol{J}}_{\varphi {\rm v}}^{\rm T} (\hat{\boldsymbol{J}}_{\varphi {\rm v}} \hat{\boldsymbol{J}}_{\varphi {\rm v}}^{\rm T})^{-1}$, $\boldsymbol{K}_{\rm ph}, \boldsymbol{K}_{\rm vh}, \boldsymbol{K}_{\rm ih} \in \mathbb{R}^{n\times n}$ *are symmetric positive definite matrices,* $k_{\rm Fh}, k_{\rm Fih} > 0$, *and*

$$\Delta\lambda_{\rm vd} \triangleq \lambda_{\rm v} - \lambda_{\rm sd} \tag{35}$$

$$\Delta F_{\rm vd} \triangleq \int_{t_0}^{t} \Delta\lambda_{\rm vd} \, {\rm d}\vartheta \,. \tag{36}$$

□

Remark 1: The position *PID* and force *PI* control law assumption for the human operator behaviour (34) is justified in Rodríguez-Angeles et al. (2015). The only difference in this work is that the directions of constrained and unconstrained motion are in terms of the virtual surface, which in turn emerges from the remote surface estimation. Notice that even if this estimation is not accurate, the master robot would generate a resistance to motion in the $\hat{\boldsymbol{J}}_{\varphi {\rm v}}$ direction. As a result, the operator feels natural to plan the position trajectory over the plane projected by $\hat{\boldsymbol{Q}}_{\rm v}$. □

Finally, it is enunciated the following useful fact taken from Rivera-Dueñas and Arteaga-Pérez (2013).

Fact 1: Assume that $\boldsymbol{q}_{\rm sd}(t)$ satisfies $\boldsymbol{\varphi}(\boldsymbol{q}_{\rm sd}) = \boldsymbol{0}$. Whenever the manipulator is restricted to fulfil (3) and the tracking error is sufficiently small, the following approximation can be made

$$\boldsymbol{e}_{\rm s} = \boldsymbol{Q}_{\rm s}(\boldsymbol{q}_{\rm s})\boldsymbol{e}_{\rm s}\,, \tag{37}$$

because the error tends to be contained in the tangent space at the point $\boldsymbol{q}_{\rm s}$, projected by $\boldsymbol{Q}_{\rm s}(\boldsymbol{q}_{\rm s})$. Furthermore, from Property 1 it follows

$$\dot{\boldsymbol{q}}_{\rm sd} \approx \boldsymbol{Q}_{\rm s}(\boldsymbol{q}_{\rm s})\dot{\boldsymbol{q}}_{\rm sd} \implies \dot{\boldsymbol{e}}_{\rm s} = \boldsymbol{Q}_{\rm s}(\boldsymbol{q}_{\rm s})\left(\dot{\boldsymbol{q}}_{\rm s} - \dot{\boldsymbol{q}}_{\rm sd}\right) \approx \boldsymbol{Q}_{\rm s}(\boldsymbol{q}_{\rm s})\dot{\boldsymbol{e}}_{\rm s}\,. \tag{38}$$

□

3.2 *Closed–loop dynamics*

Let $\tilde{\boldsymbol{q}}_{\rm s2} \triangleq \boldsymbol{q}_{\rm s2} - \hat{\boldsymbol{q}}_{s2}$ and $\tilde{\boldsymbol{z}}_i \triangleq \boldsymbol{z}_i - \hat{\boldsymbol{z}}_i, i = 1, \ldots, p$. The slave manipulator dynamics (7)–(8) and (10)–(12) in closed loop with the observer (13)–(17) results in the estimation error dynamics

$$\begin{aligned}
\dot{\tilde{\boldsymbol{q}}}_{\rm s} &= \tilde{\boldsymbol{q}}_{\rm s2} - \boldsymbol{\lambda}_{p+1}\tilde{\boldsymbol{q}}_{\rm s} && (39)\\
\dot{\tilde{\boldsymbol{q}}}_{\rm s2} &= -\boldsymbol{H}_{\rm s}^{-1}(\boldsymbol{q}_{\rm s})\left(\boldsymbol{N}_{\rm s}(\boldsymbol{q}_{\rm s}, \boldsymbol{q}_{\rm s2}) - \boldsymbol{N}_{\rm s}(\boldsymbol{q}_{\rm s}, \hat{\boldsymbol{q}}_{\rm s2})\right) + \tilde{\boldsymbol{z}}_1 - \boldsymbol{\lambda}_p\tilde{\boldsymbol{q}}_{\rm s} && (40)\\
\dot{\tilde{\boldsymbol{z}}}_1 &= \tilde{\boldsymbol{z}}_2 - \boldsymbol{\lambda}_{p-1}\tilde{\boldsymbol{q}}_{\rm s} && (41)\\
&\vdots\\
\dot{\tilde{\boldsymbol{z}}}_{p-1} &= \tilde{\boldsymbol{z}}_p - \boldsymbol{\lambda}_1\tilde{\boldsymbol{q}}_{\rm s} && (42)\\
\dot{\tilde{\boldsymbol{z}}}_p &= \boldsymbol{r}^{(p)}(t) - \boldsymbol{\lambda}_0\tilde{\boldsymbol{q}}_{\rm s}\,. && (43)
\end{aligned}$$

From the first two equations it can be obtained

$$\ddot{\tilde{\boldsymbol{q}}}_{\rm s} + \boldsymbol{\lambda}_{p+1}\dot{\tilde{\boldsymbol{q}}}_{\rm s} + \boldsymbol{\lambda}_p\tilde{\boldsymbol{q}}_{\rm s} = \tilde{\boldsymbol{z}}_1 + \boldsymbol{f}_{\rm s}(t)\,, \tag{44}$$

where

$$\begin{aligned}
\boldsymbol{f}_{\rm s}(t) \triangleq -\boldsymbol{H}_{\rm s}^{-1}(\boldsymbol{q}_{\rm s})\Big[&2\boldsymbol{C}_{\rm s}(\boldsymbol{q}_{\rm s}, \dot{\boldsymbol{e}}_{\rm s} + \dot{\boldsymbol{q}}_{\rm sd})(\dot{\tilde{\boldsymbol{q}}}_{\rm s} + \boldsymbol{\lambda}_{p+1}\tilde{\boldsymbol{q}}_{\rm s})\\
&-\boldsymbol{C}_{\rm s}(\boldsymbol{q}_{\rm s}, \dot{\tilde{\boldsymbol{q}}}_{\rm s} + \boldsymbol{\lambda}_{p+1}\tilde{\boldsymbol{q}}_{\rm s})(\dot{\tilde{\boldsymbol{q}}}_{\rm s} + \boldsymbol{\lambda}_{p+1}\tilde{\boldsymbol{q}}_{\rm s}) + \boldsymbol{D}_{\rm s}(\dot{\tilde{\boldsymbol{q}}}_{\rm s} + \boldsymbol{\lambda}_{p+1}\tilde{\boldsymbol{q}}_{\rm s})\Big]\,.
\end{aligned} \tag{45}$$

By taking p time derivatives of (44) and after (41)–(43) one has

$$\tilde{\boldsymbol{q}}_{\rm s}^{(p+2)} + \boldsymbol{\lambda}_{p+1}\tilde{\boldsymbol{q}}_{\rm s}^{(p+1)} + \cdots + \boldsymbol{\lambda}_0\tilde{\boldsymbol{q}}_{\rm s} = \boldsymbol{r}^{(p)}(t) + \boldsymbol{f}_{\rm s}^{(p)}(t)\,. \tag{46}$$

This last equation can be rewritten in state space form as

$$\dot{\boldsymbol{x}}_{\rm o} = \boldsymbol{A}\boldsymbol{x}_{\rm o} + \boldsymbol{B}\boldsymbol{r}_{\rm f}\,, \tag{47}$$

where $\boldsymbol{r}_\mathrm{f} = \boldsymbol{r}^{(p)}(t) + \boldsymbol{f}_\mathrm{s}^{(p)}(t)$ and

$$\boldsymbol{x}_\mathrm{o} \triangleq \begin{bmatrix} \tilde{\boldsymbol{q}}_\mathrm{s} & \cdots & \tilde{\boldsymbol{q}}_\mathrm{s}^{(p+1)} \end{bmatrix}^\mathrm{T} \tag{48}$$

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{O} & \boldsymbol{I} & \cdots & \boldsymbol{O} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{I} \\ -\boldsymbol{\lambda}_0 & -\boldsymbol{\lambda}_1 & \cdots & -\boldsymbol{\lambda}_{p+1} \end{bmatrix} \tag{49}$$

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{I} \end{bmatrix}^\mathrm{T} . \tag{50}$$

In order to obtain the closed loop dynamics for the slave manipulator, first the control law (23) is rewritten as

$$\begin{aligned} \boldsymbol{\tau}_\mathrm{s} = &-\boldsymbol{K}_\mathrm{vs}\dot{\boldsymbol{e}}_\mathrm{s} - \boldsymbol{K}_\mathrm{ps}\boldsymbol{e}_\mathrm{s} - \boldsymbol{K}_\mathrm{is}\boldsymbol{Q}_\mathrm{s}\int_0^t \boldsymbol{e}_\mathrm{s}\,\mathrm{d}\vartheta - \boldsymbol{J}_{\varphi\mathrm{s}}^\mathrm{T}\lambda_\mathrm{sd} + k_\mathrm{Fis}\boldsymbol{J}_{\varphi\mathrm{s}}^{+}\Delta\bar{F}_\mathrm{s} \\ &+ \boldsymbol{K}_\mathrm{vs}(\dot{\tilde{\boldsymbol{q}}}_\mathrm{s} + \boldsymbol{\lambda}_{p+1}\tilde{\boldsymbol{q}}_\mathrm{s}) + \boldsymbol{K}_\mathrm{is}\tilde{\boldsymbol{Q}}_\mathrm{s}\int_0^t \boldsymbol{e}_\mathrm{s}\,\mathrm{d}\vartheta + \tilde{\boldsymbol{J}}_{\varphi s}^\mathrm{T}\lambda_\mathrm{sd} - k_\mathrm{Fis}\tilde{\boldsymbol{J}}_{\varphi s}^{+}\Delta\bar{F}_\mathrm{s} , \end{aligned} \tag{51}$$

where $\tilde{\boldsymbol{Q}}_\mathrm{s} \triangleq \boldsymbol{Q}_\mathrm{s} - \hat{\boldsymbol{Q}}_\mathrm{s}$, $\tilde{\boldsymbol{J}}_{\varphi s} \triangleq \boldsymbol{J}_{\varphi s} - \hat{\boldsymbol{J}}_{\varphi s}$, and $\bar{\boldsymbol{J}}_{\varphi s}^{+} \triangleq \boldsymbol{J}_{\varphi s}^{+} - \hat{\boldsymbol{J}}_{\varphi s}^{+}$. Define

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\sigma} \triangleq \dot{\boldsymbol{e}}_\mathrm{s} + \boldsymbol{\Lambda}\boldsymbol{e}_\mathrm{s} , \tag{52}$$

where $\boldsymbol{\Lambda} \in \mathbb{R}^{n\times n}$ is a diagonal positive definite matrix. It is always possible to find $\boldsymbol{K}_\mathrm{vs} \in \mathbb{R}^{n\times n}$ and $\bar{\boldsymbol{K}}_\mathrm{is} \in \mathbb{R}^{n\times n}$ such that

$$\boldsymbol{K}_\mathrm{ps} = \boldsymbol{K}_\mathrm{vs}\boldsymbol{\Lambda} + \bar{\boldsymbol{K}}_\mathrm{is} \tag{53}$$

$$\boldsymbol{K}_\mathrm{is} = \bar{\boldsymbol{K}}_\mathrm{is}\boldsymbol{\Lambda} . \tag{54}$$

Therefore, the control law (51) is equivalent to

$$\begin{aligned} \boldsymbol{\tau}_\mathrm{s} = &-\boldsymbol{K}_\mathrm{vs}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\sigma} - \bar{\boldsymbol{K}}_\mathrm{is}\boldsymbol{Q}_\mathrm{s}\boldsymbol{\sigma} - \boldsymbol{J}_{\varphi\mathrm{s}}^\mathrm{T}\lambda_\mathrm{sd} + k_\mathrm{Fis}\boldsymbol{J}_{\varphi\mathrm{s}}^{+}\Delta\bar{F}_\mathrm{s} + \boldsymbol{K}_\mathrm{vs}(\dot{\tilde{\boldsymbol{q}}}_\mathrm{s} + \boldsymbol{\lambda}_{p+1}\tilde{\boldsymbol{q}}_\mathrm{s}) \\ &+ \boldsymbol{K}_\mathrm{is}\tilde{\boldsymbol{Q}}_\mathrm{s}\int_0^t \boldsymbol{e}_\mathrm{s}\,\mathrm{d}\vartheta + \tilde{\boldsymbol{J}}_{\varphi s}^\mathrm{T}\lambda_\mathrm{sd} - k_\mathrm{Fis}\tilde{\boldsymbol{J}}_{\varphi s}^{+}\Delta\bar{F}_\mathrm{s} , \end{aligned} \tag{55}$$

as long as $\boldsymbol{Q}_\mathrm{s}\boldsymbol{\Lambda} = \boldsymbol{\Lambda}\boldsymbol{Q}_\mathrm{s}$ holds, what can be achieved for instance by setting $\boldsymbol{\Lambda} = k_\lambda\boldsymbol{I}$. It is also defined

$$\dot{\boldsymbol{q}}_\mathrm{r} \triangleq \dot{\boldsymbol{q}}_\mathrm{sd} - \boldsymbol{\Lambda}\boldsymbol{e}_\mathrm{s} - \boldsymbol{K}_\mathrm{vs}^{-1}\bar{\boldsymbol{K}}_\mathrm{is}\boldsymbol{Q}_\mathrm{s}\boldsymbol{\sigma} + \frac{1}{2}\boldsymbol{K}_\mathrm{vs}^{-1}\boldsymbol{J}_{\varphi s}^\mathrm{T}\Delta\lambda_\mathrm{s} + \frac{1}{2}k_\mathrm{Fis}\boldsymbol{K}_\mathrm{vs}^{-1}\boldsymbol{J}_{\varphi s}^{+}\Delta\bar{F}_\mathrm{s} \tag{56}$$

$$\begin{aligned} \boldsymbol{s} \triangleq \dot{\boldsymbol{q}}_\mathrm{s} - \dot{\boldsymbol{q}}_\mathrm{r} &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\sigma} + \boldsymbol{K}_\mathrm{vs}^{-1}\bar{\boldsymbol{K}}_\mathrm{is}\boldsymbol{Q}_\mathrm{s}\boldsymbol{\sigma}\right) + \left(-\frac{1}{2}\boldsymbol{K}_\mathrm{vs}^{-1}\boldsymbol{J}_{\varphi s}^\mathrm{T}\Delta\lambda_\mathrm{s} - \frac{1}{2}k_\mathrm{Fis}\boldsymbol{K}_\mathrm{vs}^{-1}\boldsymbol{J}_{\varphi s}^{+}\Delta\bar{F}_\mathrm{s}\right) \\ &= \boldsymbol{s}_\mathrm{p} + \boldsymbol{s}_\mathrm{F} , \end{aligned} \tag{57}$$

where $\Delta\lambda_\mathrm{s} \triangleq \lambda_\mathrm{s} - \lambda_\mathrm{sd}$ is the force tracking error. The closed loop dynamics of the slave manipulator is then described by

$$\boldsymbol{H}_\mathrm{s}\dot{\boldsymbol{s}} + \boldsymbol{C}_\mathrm{s}\boldsymbol{s} + \boldsymbol{K}_\mathrm{Dvs}\boldsymbol{s} = \frac{1}{2}\boldsymbol{J}_{\varphi s}^\mathrm{T}\Delta\lambda_\mathrm{s} + \frac{1}{2}k_\mathrm{Fis}\boldsymbol{J}_{\varphi s}^{+}\Delta\bar{F}_\mathrm{s} + \boldsymbol{y}_\mathrm{a} , \tag{58}$$

where $\boldsymbol{K}_{\mathrm{Dvs}} = \boldsymbol{K}_{\mathrm{vs}} + \boldsymbol{D}_{\mathrm{s}}$, and

$$\begin{aligned}\boldsymbol{y}_{\mathrm{a}} = \boldsymbol{K}_{\mathrm{vs}}(\dot{\tilde{\boldsymbol{q}}}_{\mathrm{s}} + \boldsymbol{\lambda}_{p+1}\tilde{\boldsymbol{q}}_{\mathrm{s}}) + \boldsymbol{K}_{\mathrm{is}}\tilde{\boldsymbol{Q}}_{\mathrm{s}}\int_0^t \boldsymbol{e}_{\mathrm{s}}\,\mathrm{d}\vartheta + \tilde{\boldsymbol{J}}_{\varphi s}^{\mathrm{T}}\lambda_{\mathrm{sd}} - k_{\mathrm{Fis}}\tilde{\boldsymbol{J}}_{\varphi s}^{+}\Delta\bar{F}_{\mathrm{s}} \\ - \left(\boldsymbol{H}_{\mathrm{s}}(\boldsymbol{q}_{\mathrm{s}})\ddot{\boldsymbol{q}}_{\mathrm{r}} + \boldsymbol{C}_{\mathrm{s}}(\boldsymbol{q}_{\mathrm{s}}, \boldsymbol{q}_{\mathrm{s2}})\dot{\boldsymbol{q}}_{\mathrm{r}} + \boldsymbol{D}_{\mathrm{s}}\dot{\boldsymbol{q}}_{\mathrm{r}} + \boldsymbol{g}(\boldsymbol{q}_{\mathrm{s}})\right) . \end{aligned} \tag{59}$$

To obtain the dynamics of the surface estimation error, let

$$\tilde{\boldsymbol{J}}_{\varphi\mathrm{xs}} \triangleq \boldsymbol{J}_{\varphi\mathrm{xs}} - \hat{\boldsymbol{J}}_{\varphi\mathrm{xs}} . \tag{60}$$

By taking into account (19) it is

$$\dot{\tilde{\boldsymbol{J}}}_{\varphi\mathrm{xs}}^{\mathrm{T}} = \dot{\boldsymbol{J}}_{\varphi\mathrm{xs}}^{\mathrm{T}} - \dot{\hat{\boldsymbol{J}}}_{\varphi\mathrm{xs}}^{\mathrm{T}} = \dot{\boldsymbol{J}}_{\varphi\mathrm{xs}}^{\mathrm{T}} - \left(\frac{\gamma}{\hat{\lambda}_{\mathrm{s}} + \epsilon}\right)\hat{\boldsymbol{Q}}_{\mathrm{xs}}\boldsymbol{J}_{\mathrm{s}}^{-\mathrm{T}}(\boldsymbol{q}_{\mathrm{s}})\boldsymbol{H}_{\mathrm{s}}(\boldsymbol{q}_{\mathrm{s}})\hat{\boldsymbol{z}}_1 . \tag{61}$$

Finally, for the master robot, from (1), (31) and (34), one obtains

$$\begin{aligned}\hat{\boldsymbol{J}}_{\varphi\mathrm{v}}^{\mathrm{T}}\left\{k_{\mathrm{Fv}}\Delta\lambda_{\mathrm{vs}} + k_{\mathrm{Fiv}}\Delta F_{\mathrm{vs}} + k_{\mathrm{Fh}}\Delta\lambda_{\mathrm{vd}} + k_{\mathrm{Fih}}\Delta F_{\mathrm{vd}}\right\} \\ + \hat{\boldsymbol{Q}}_{\mathrm{v}}\left\{\boldsymbol{K}_{\mathrm{ph}}\boldsymbol{e}_{\mathrm{m}} + \boldsymbol{K}_{\mathrm{vh}}\dot{\boldsymbol{e}}_{\mathrm{m}} + \boldsymbol{K}_{\mathrm{ih}}\int_0^t \boldsymbol{e}_{\mathrm{m}}\,\mathrm{d}\vartheta\right\} = \boldsymbol{0} .\end{aligned} \tag{62}$$

From (24)–(25) and (32)–(33) it is easy to get

$$\Delta\lambda_{\mathrm{vs}} = \Delta\lambda_{\mathrm{vd}} - \Delta\bar{\lambda}_{\mathrm{s}} \tag{63}$$

$$\Delta F_{\mathrm{vs}} = \Delta F_{\mathrm{vd}} - \Delta\bar{F}_{\mathrm{s}} , \tag{64}$$

so that (62) can be rewritten as

$$\begin{aligned}\hat{\boldsymbol{J}}_{\varphi\mathrm{v}}^{\mathrm{T}}\left\{(k_{\mathrm{Fv}} + k_{\mathrm{Fh}})\Delta\lambda_{\mathrm{vd}} + (k_{\mathrm{Fiv}} + k_{\mathrm{Fih}})\Delta F_{\mathrm{vd}}\right\} \\ + \hat{\boldsymbol{Q}}_{\mathrm{v}}\left\{\boldsymbol{K}_{\mathrm{ph}}\boldsymbol{e}_{\mathrm{m}} + \boldsymbol{K}_{\mathrm{vh}}\dot{\boldsymbol{e}}_{\mathrm{m}} + \boldsymbol{K}_{\mathrm{ih}}\int_0^t \boldsymbol{e}_{\mathrm{m}}\,\mathrm{d}\vartheta\right\} = \hat{\boldsymbol{J}}_{\varphi\mathrm{v}}^{\mathrm{T}}\left\{k_{\mathrm{Fv}}\Delta\bar{\lambda}_{\mathrm{s}} + k_{\mathrm{Fiv}}\Delta\bar{F}_{\mathrm{s}}\right\} .\end{aligned} \tag{65}$$

Notice that (65) describes two dynamics evolving in orthogonal subspaces. By taking advantage of the fact that $\hat{\boldsymbol{J}}_{\varphi\mathrm{v}}$ is full rank, they can be analysed separately as

$$(k_{\mathrm{Fv}} + k_{\mathrm{Fh}})\Delta\lambda_{\mathrm{vd}} + (k_{\mathrm{Fiv}} + k_{\mathrm{Fih}})\Delta F_{\mathrm{vd}} = k_{\mathrm{Fv}}\Delta\bar{\lambda}_{\mathrm{s}} + k_{\mathrm{Fiv}}\Delta\bar{F}_{\mathrm{s}} \tag{66}$$

$$\hat{\boldsymbol{Q}}_{\mathrm{v}}\left(\boldsymbol{K}_{\mathrm{vh}}\dot{\boldsymbol{e}}_{\mathrm{m}} + \boldsymbol{K}_{\mathrm{ph}}\boldsymbol{e}_{\mathrm{m}} + \boldsymbol{K}_{\mathrm{ih}}\int_0^t \boldsymbol{e}_{\mathrm{m}}\,\mathrm{d}\vartheta\right) = \boldsymbol{0} . \tag{67}$$

Before stating the main result on the master–slave teleoperation system, an auxiliary result is presented, which only takes into account the slave manipulator in closed loop with the force and velocity observer and the surface estimator.

Theorem 1 (Gutiérrez-Giles and Arteaga-Pérez (2016))**:** *Consider the slave manipulator in contact with a rigid surface described by (2)–(3) in closed loop with the observer (13)–(17) and (18), the controller (23), and the surface estimator (19), whose complete closed loop dynamics is given* *by (24)–(25), (47), (58), and (61). Suppose that* $\boldsymbol{q}_{\mathrm{sd}}(t)$ *is smooth. Let*

$$\boldsymbol{y}_{\mathrm{s}} \triangleq \begin{bmatrix} \boldsymbol{x}_{\mathrm{o}} & \boldsymbol{s} & \Delta\bar{F}_{\mathrm{s}} & \tilde{\boldsymbol{J}}_{\varphi\mathrm{xs}} \end{bmatrix}^{\mathrm{T}}, \tag{68}$$

and define a region $\mathcal{D}_{\mathrm{s}} \triangleq \{\boldsymbol{y}_{\mathrm{s}} \in \mathbb{R}^{(p+2)n+4} \mid \|\boldsymbol{y}_{\mathrm{s}}\| \leq y_{\max}\}$ *where* $y_{\max}$ *is a positive constant small enough for Fact 1 to hold. Assume that the manipulator never loses contact with the environment. Then, a set of controller gains* $\boldsymbol{K}_{\mathrm{ps}}$, $\boldsymbol{K}_{\mathrm{vs}}$, $\boldsymbol{K}_{\mathrm{is}}$ *and* k_{Fis} *in (23),* γ *in (19), and a set of observer gains* $\boldsymbol{\lambda}_0, \ldots, \boldsymbol{\lambda}_{p+1}$ *in (49) can always be found to achieve ultimate boundedness of* $\boldsymbol{y}_{\mathrm{s}}$ *provided the initial condition* $\boldsymbol{y}_{\mathrm{s}}(t_0)$ *is small enough such that* $\boldsymbol{y}_{\mathrm{s}}$ *does never leave* $\mathcal{D}_{\mathrm{s}}$ *during the transient response. Furthermore, the tracking errors* $\mathbf{e}_{\mathrm{s}}$, $\dot{\mathbf{e}}_{\mathrm{s}}$, *and* $\Delta\lambda_{\mathrm{s}}$, *and the estimation errors* $\boldsymbol{x}_{\mathrm{o}}$ *and* $\tilde{\boldsymbol{J}}_{\varphi\mathrm{xs}}$ *can be made arbitrarily small as well.* □

Proof. See Appendix A. □

Now, the main result of this work is presented , which is focused on the ultimate boundedness of the state and the transparency of the teleoperation system.

Theorem 2: *Let the master–slave teleoperation system described by (1)–(2) be in closed loop with the force and velocity observers (13)–(17) and (18), the surface estimator (19), and let the behaviour of the human operator be described as in Assumption 3. Then, the system error dynamics is completely characterised by (24)–(25), (35)–(36), (47), (58), (61), and (65). Let*

$$\boldsymbol{y} \triangleq \begin{bmatrix} \boldsymbol{x}_{\mathrm{o}} & \boldsymbol{s} & \Delta\bar{F}_{\mathrm{s}} & \tilde{\boldsymbol{J}}_{\varphi\mathrm{xs}} & \Delta F_{\mathrm{vd}} \end{bmatrix}^{\mathrm{T}}, \tag{69}$$

and define a region $\mathcal{D} \triangleq \{\boldsymbol{y} \in \mathbb{R}^{(p+2)n+5} \mid \|\boldsymbol{y}\| \leq y_{\max}\}$ *where* $y_{\max}$ *is a positive constant small enough for Fact 1 to hold. Assume that the manipulator never loses contact with the environment. Furthermore, assume that the desired position for the master manipulator* $\boldsymbol{q}_{\mathrm{md}}(t)$, *planned by the operator, is such that the approximations (analogous to Fact 1)*

$$\hat{\boldsymbol{Q}}_{\mathrm{v}}\boldsymbol{e}_{\mathrm{m}} \approx \boldsymbol{e}_{\mathrm{m}} \tag{70}$$

$$\hat{\boldsymbol{Q}}_{\mathrm{v}}\dot{\boldsymbol{e}}_{\mathrm{m}} \approx \dot{\boldsymbol{e}}_{\mathrm{m}} \tag{71}$$

hold. Then, a set of controller gains $\boldsymbol{K}_{\mathrm{ps}}$, $\boldsymbol{K}_{\mathrm{vs}}$, $\boldsymbol{K}_{\mathrm{is}}$ *and* k_{Fis} *in (23),* k_{Fv} *and* k_{Fiv} *in (31),* γ *in (19), and a set of observer gains* $\boldsymbol{\lambda}_0, \ldots, \boldsymbol{\lambda}_{p+1}$ *in (49) can be found to achieve: (i) position and velocity tracking at the master side, (ii) ultimate boundedness of* $\boldsymbol{y}$, *with arbitrary small ultimate bound, and (iii) transparency of the teleoperation system.* □

Proof. (*i*) For simplicity's sake let $\boldsymbol{K}_{\mathrm{vh}} = k_{\mathrm{vh}}\boldsymbol{I}_{n\times n}$ and $\boldsymbol{K}_{\mathrm{ih}} = k_{\mathrm{ih}}\boldsymbol{I}_{n\times n}$. Define $\boldsymbol{e}_{\mathrm{Im}} \triangleq \int_{t_0}^{t} \boldsymbol{e}_{\mathrm{m}}\, \mathrm{d}\vartheta$ and let

$$V_{\mathrm{m}} = \frac{1}{2}k_{\mathrm{vh}}\boldsymbol{e}_{\mathrm{m}}^{\mathrm{T}}\boldsymbol{e}_{\mathrm{m}} + \frac{1}{2}k_{\mathrm{ih}}\boldsymbol{e}_{\mathrm{Im}}^{\mathrm{T}}\boldsymbol{e}_{\mathrm{Im}}. \tag{72}$$

By taking into account (70)–(71), the time derivative of (72) along (67) is given by

$$\dot{V}_{\mathrm{m}} = -\boldsymbol{e}_{\mathrm{m}}^{\mathrm{T}}\boldsymbol{K}_{\mathrm{ph}}\boldsymbol{e}_{\mathrm{m}} \leq 0. \tag{73}$$

From LaSalle's theorem if $\dot{V}_{\mathrm{m}} \equiv 0 \implies \boldsymbol{e}_{\mathrm{m}} \equiv \boldsymbol{0} \implies \dot{\boldsymbol{e}}_{\mathrm{m}} \equiv \boldsymbol{0}$ then the trajectories of (67) tend asymptotically to the set $(\dot{\boldsymbol{e}}_{\mathrm{m}}, \boldsymbol{e}_{\mathrm{m}}, \hat{\boldsymbol{Q}}_{\mathrm{v}}\boldsymbol{e}_{\mathrm{Im}}) = (\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0})$[1].

[1]It is important to point out that the dynamics described in (67) is actually independent of the rest of the system, because no matter whether the estimate of $\hat{\boldsymbol{Q}}_{\mathrm{v}}$ is accurate or not, the master controller will create that surface for the person.

(*ii*) Notice that all the premises of Theorem 1 are satisfied, since $\boldsymbol{y}_{\mathrm{s}}$ in (68) is a subset of $\boldsymbol{y}$ in (69) and then it must be bounded in $\mathcal{D}$. Some slight modifications must be done to extend the proof of Theorem 1 for the state $\boldsymbol{y}$ defined in (69). First, in part a) of the proof of Theorem 1 it is shown that all signals of interest are bounded in $\mathcal{D}$. Trivially ΔF_{vd} is bounded in $\mathcal{D}$ and, under the arguments of the original proof, $\Delta\bar{\lambda}_{\mathrm{s}}$ and $\Delta\bar{F}_{\mathrm{s}}$ are also bounded, so that after (66), $\Delta\lambda_{\mathrm{vd}}$ must be bounded as well. Second, in part c) of the proof of Theorem 1 it is proposed the function

$$V = \boldsymbol{x}_{\mathrm{o}}^{\mathrm{T}}\boldsymbol{P}_{\mathrm{o}}\boldsymbol{x}_{\mathrm{o}} + \frac{1}{2}\boldsymbol{s}^{\mathrm{T}}\boldsymbol{H}_{\mathrm{s}}(\boldsymbol{q})\boldsymbol{s} + \frac{1}{4}\frac{k_{\mathrm{Fis}}}{k_{\mathrm{vs}}}(\Delta\bar{F}_{\mathrm{s}})^2 + \frac{1}{2}\tilde{\boldsymbol{J}}_{\varphi\mathrm{xs}}\tilde{\boldsymbol{J}}_{\varphi\mathrm{xs}}^{\mathrm{T}}\,, \tag{74}$$

with $\boldsymbol{P}_{\mathrm{o}} = \boldsymbol{P}_{\mathrm{o}}^{\mathrm{T}} > \boldsymbol{O}$ the solution of

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}_{\mathrm{o}} + \boldsymbol{P}_{\mathrm{o}}\boldsymbol{A} = -\boldsymbol{Q}_{\mathrm{o}}\,, \tag{75}$$

where $\boldsymbol{Q}_{\mathrm{o}}$ is a positive definite matrix and $\boldsymbol{A}$ is given by (49). To include the state ΔF_{vd}, it is proposed

$$V_{\mathrm{F}} = \frac{1}{2}\frac{k_{\mathrm{Fv}} + k_{\mathrm{Fh}}}{k_{\mathrm{Fiv}} + k_{\mathrm{Fih}}}\left(\Delta F_{\mathrm{vd}}\right)^2\,, \tag{76}$$

whose time derivative along (66) is given by

$$\dot{V}_{\mathrm{F}} = -\left(\Delta F_{\mathrm{vd}}\right)^2 + \left(\Delta F_{\mathrm{vd}}\right)\frac{k_{\mathrm{Fv}}\Delta\bar{\lambda}_{\mathrm{s}} + k_{\mathrm{Fiv}}\Delta\bar{F}_{\mathrm{s}}}{k_{\mathrm{Fiv}} + k_{\mathrm{Fih}}} \tag{77}$$

$$\leq -|\Delta F_{\mathrm{vd}}|\left(|\Delta F_{\mathrm{vd}}| - \frac{|k_{\mathrm{Fv}}\Delta\bar{\lambda}_{\mathrm{s}} + k_{\mathrm{Fiv}}\Delta\bar{F}_{\mathrm{s}}|}{k_{\mathrm{Fiv}} + k_{\mathrm{Fih}}}\right)\,. \tag{78}$$

By setting k_{Fiv} sufficiently large, the term $(|k_{\mathrm{Fv}}\Delta\bar{\lambda}_{\mathrm{s}} + k_{\mathrm{Fiv}}\Delta\bar{F}_{\mathrm{s}}|)/(k_{\mathrm{Fiv}} + k_{\mathrm{Fih}})$ in (78) can be made arbitrarily small. On the other hand, by adding the functions (74) and (76) one obtains

$$V_{\mathrm{T}} = V + V_{\mathrm{F}} = \boldsymbol{y}^{\mathrm{T}}\begin{bmatrix} \boldsymbol{P}_{\mathrm{o}} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \frac{1}{2}\boldsymbol{H}_{\mathrm{s}}(\boldsymbol{q}_{\mathrm{s}}) & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \frac{1}{4}k_{\mathrm{Fis}}/k_{\mathrm{vs}} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \frac{1}{2}\boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \frac{1}{2}k_{\mathrm{FT}} \end{bmatrix}\boldsymbol{y} = \boldsymbol{y}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{q}_{\mathrm{s}})\boldsymbol{y}\,, \tag{79}$$

where $k_{\mathrm{FT}} = (k_{\mathrm{Fv}} + k_{\mathrm{Fh}})/(k_{\mathrm{Fiv}} + k_{\mathrm{Fih}})$. After Property 2, one can find two positive constants, λ_{m} and λ_{M}, such that

$$\lambda_{\mathrm{m}}\|\boldsymbol{y}\|^2 \leq V_{\mathrm{T}}(\boldsymbol{y}) \leq \lambda_{\mathrm{M}}\|\boldsymbol{y}\|^2\,. \tag{80}$$

Accordingly with the above discussion and from the proof of Theorem 1, an arbitrarily small positive constant μ_{T} can always be found, such that [2]

$$\dot{V}_{\mathrm{T}} \leq 0 \quad \text{if} \quad \|\boldsymbol{y}\| \geq \mu_{\mathrm{T}}\,. \tag{81}$$

Once $\|\boldsymbol{y}\| = \mu_{\mathrm{T}}$, from (80) the maximum value that $\|\boldsymbol{y}\|$ can take is given by

$$\lambda_{\mathrm{m}}\|\boldsymbol{y}\|^2 \leq V_{\mathrm{T}}(\boldsymbol{y}) \leq \lambda_{\mathrm{M}}\mu_{\mathrm{T}}^2 \implies \|\boldsymbol{y}\| \leq \sqrt{\frac{\lambda_{\mathrm{M}}}{\lambda_{\mathrm{m}}}}\mu_{\mathrm{T}} \triangleq b_{\mathrm{T}}\,, \tag{82}$$

[2] Notice that we are simplifying notation since λ_{m} and λ_{M} in (80) are not the same as those given in (A46).

where b_{T} is the ultimate bound of the state $\boldsymbol{y}$, that can be made arbitrarily small by an appropriate selection of the controllers and the observer gains. Recall that it must be guaranteed that $\|\boldsymbol{y}\| \leq y_{\max}$, $\forall t \geq t_0$. This can be done by setting gains large enough to satisfy

$$\mu_{\mathrm{T}} < \sqrt{\frac{\lambda_{\mathrm{m}}}{\lambda_{\mathrm{M}}}} y_{\max} \,. \tag{83}$$

Also, the initial condition must satisfy

$$\|\boldsymbol{y}(t_0)\| < \sqrt{\frac{\lambda_{\mathrm{m}}}{\lambda_{\mathrm{M}}}} y_{\max} \tag{84}$$

to guarantee that $\boldsymbol{y}$ never leaves the region $\mathcal{D}$.

(*iii*) Notice that after (35)–(36) and the above discussion, (66) represents a stable filter with an arbitrarily small input. This implies, after (63)–(64) that

$$(\Delta\bar{\lambda}_{\mathrm{s}}, \Delta\bar{F}_{\mathrm{s}}) \approx (0,0) \implies (\Delta\lambda_{\mathrm{vd}}, \Delta F_{\mathrm{vd}}) \approx 0 \implies \lambda_{\mathrm{sd}} \approx \lambda_{\mathrm{v}} \approx \lambda_{\mathrm{s}} \approx \hat{\lambda}_{\mathrm{s}}, \text{ as } t \to \infty \,, \tag{85}$$

which establishes the convergence of the observer, the force tracking, and the transparency of the teleoperation system. □

4. Simulation results

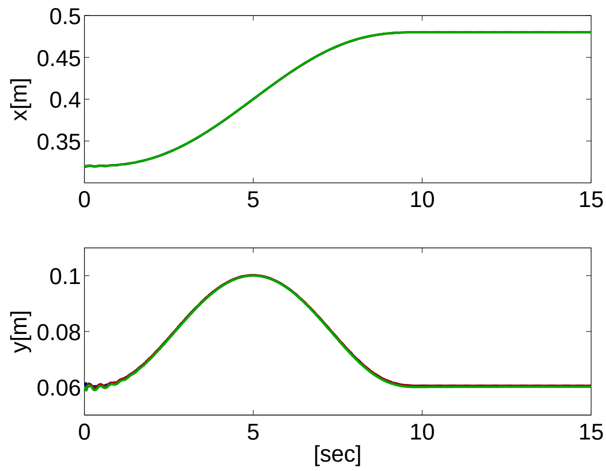

Figure 2. Position tracking in Cartesian coordinates: desired (- - -), master (—), slave (—).

To validate the approach proposed in Section 3, a numerical simulation was carried out consisting in two full–actuated revolute manipulators with two joints in planar movement. The parameters for both robots are: mass of the links $m_1 = 3.9473$[Kg], $m_2 = 0.6232$[Kg], length of the links $l_1 = l_2 = 0.38$[m], and viscous friction coefficients $d_1 = d_2 = 1.2$[Kg $\cdot$ m/sec]. The (assumed unknown) surface is a segment of a circle described by

$$\varphi_{\mathrm{s}}(\boldsymbol{x}_{\mathrm{s}}) = (x-h)^2 + (y-k)^2 - r^2 = 0 \,. \tag{86}$$

where (x, y) stands for the slave task–space coordinates, *i.e.*, $\boldsymbol{x}_{\mathrm{s}} = \begin{bmatrix} x & y \end{bmatrix}^{\mathrm{T}}$, $r = 0.1$[m] is the radius, and $(h, k) = (0.4, 0)$[m] are the coordinates of the centre of the circle. At the beginning of

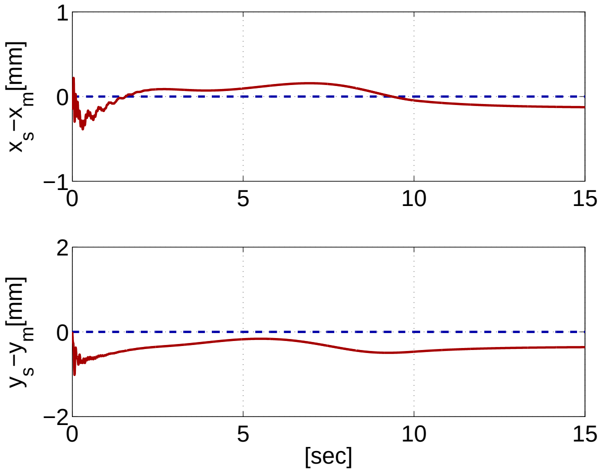

Figure 3. Position tracking error.

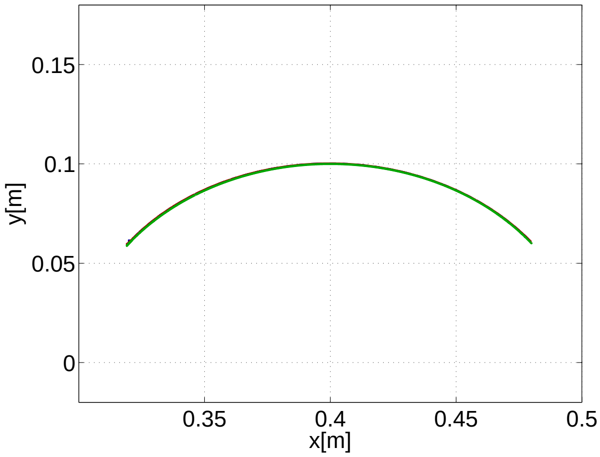

Figure 4. Position tracking in the xy plane.

the task, the tip of the slave manipulator is in contact with the surface and the Cartesian position of both manipulators coincide, but the initial condition for the estimator of the gradient (19) is set with an initial error (see Figure 9).

The task consisted in following a trajectory from the point $(x, y) = (0.32, 0.06)$[m] to the point $(x, y) = (0.48, 0.06)$[m] over the surface in $t_{\rm f} = 10$[sec], while simultaneously it is desired to track a force signal given by

$$\lambda_{\rm sd}(t) = \begin{cases} 20 + 40\,(\cos(0.8\pi t/t_{\rm f})\sin(1.6\pi t/t_{\rm f}))\ [\mathrm{N}] & \text{if } t \leq t_{\rm f} \\ 20 + 40\,(\cos(0.8\pi)\sin(1.6\pi))\ [\mathrm{N}] & \text{if } t > t_{\rm f}\,. \end{cases} \tag{87}$$

The controller gains for the slave control law (23) are $\boldsymbol{K}_{\rm ps} = {\rm diag}(2000, 2000)$, $\boldsymbol{K}_{\rm vs} = {\rm diag}(10, 10)$, $\boldsymbol{K}_{\rm is} = {\rm diag}(1000, 1000)$, and $k_{\rm Fis} = 0.5$. For the master control law (31) the gains are $k_{\rm Fv} = 0.1$ and $k_{\rm Fiv} = 0.1$. The proposed gains for the control law of Assumption 3 are $\boldsymbol{K}_{\rm ph} = {\rm diag}(100, 100)$, $\boldsymbol{K}_{\rm vh} = {\rm diag}(1, 1)$, $\boldsymbol{K}_{\rm ih} = {\rm diag}(0.1, 0.1)$, $k_{\rm Fh} = 0.01$, and $k_{\rm Fih} = 0.1$. For the observer (13)–(17) it was set $p = 2$, with the observer gains $\lambda_0 = 2.56 \times 10^6\boldsymbol{I}$, $\lambda_1 = 2.56 \times 10^5\boldsymbol{I}$, $\lambda_2 = 9600\boldsymbol{I}$, and $\lambda_0 = 160\boldsymbol{I}$, *i.e.*, the poles of the observer were located at $p_{\rm o1} = p_{\rm o2} = p_{\rm o3} = p_{\rm o4} = -40$. For the surface estimator (19) there were chosen $\gamma = 10$ and $\epsilon = 0.0001$. Also, it was set $\eta = 500$ in (27). Finally, for the Lagrange multiplier computation in (30) there were set $\alpha_{\rm v} = 1$, $\xi = 0.1$,

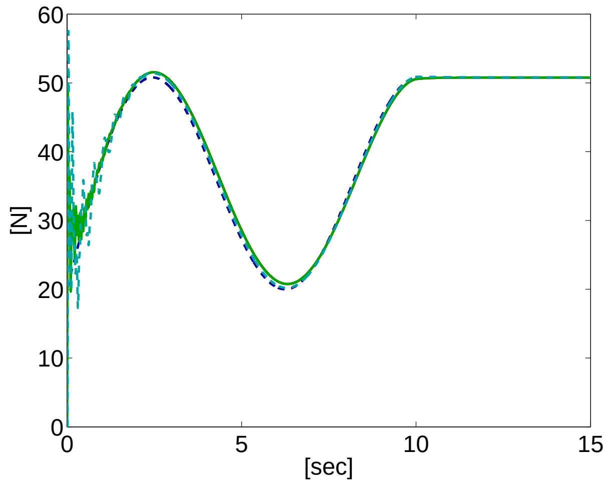

Figure 5. Force tracking and estimation: desired (- - -), real(—), estimated(—), virtual(- - -).

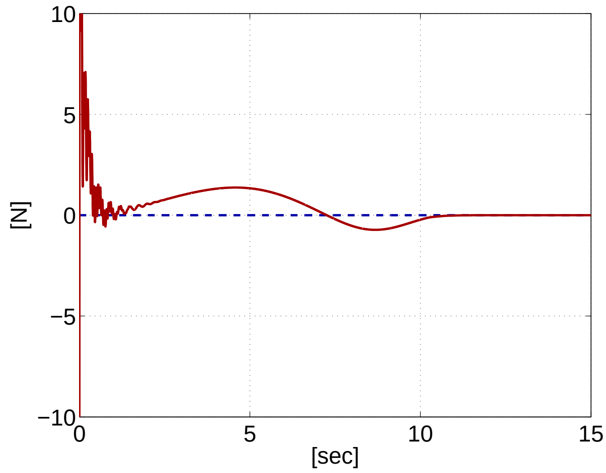

Figure 6. Force tracking error.

and $\omega_n = 200$.

In Figure 2 the position tracking in Cartesian coordinates is shown, while in Figure 3 the tracking error, $\boldsymbol{x}_s - \boldsymbol{x}_m$, is presented, where the ultimate boundedness guaranteed by the proposed scheme can be appreciated. In Figure 4 it is shown the position tracking in the xy plane. The force tracking of the signal (87) is presented in Figure 5, while the force tracking error and the force estimation error are shown in Figures 6 and 7, respectively. Finally, the estimation of the components of the surface gradient vector are displayed in Figure 8, while a zoom in the axis time of the first second is presented in Figure 9.

The simulation results clearly show that there is an ultimate bounded error for both position and force tracking of the teleoperation scheme, while in Figure 5 it can be seen that $\lambda_s \approx \lambda_v \approx \lambda_{sd}$, what shows the transparency of the system. Furthermore, in Figures 8 and 9 it is shown that the gradient of the surface can be on line estimated without force nor velocity measurements at the slave side.

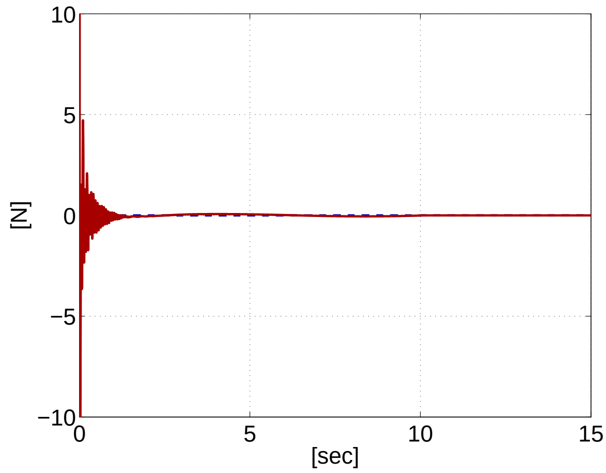

Figure 7. Force estimation error.

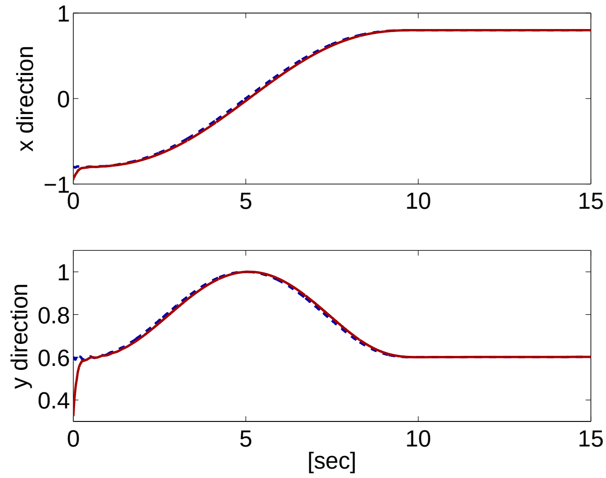

Figure 8. Estimation of the surface gradient: $\boldsymbol{J}_{\varphi \mathrm{xs}}$ (- - -), $\hat{\boldsymbol{J}}_{\varphi \mathrm{xs}}$ (—).

5. Experimental Results

6. Conclusions and future work

A master–slave teleoperation system interacting with a rigid surface, whose geometry is completely unknown was studied. To avoid force and velocity measurements at the slave side, an extended–state high–gain observer was designed, that estimates in an arbitrary close manner, the unmeasured signals. Besides, an on line estimator of the gradient of the unknown contact surface was proposed. Two control laws are designed for both the master and the slave robots to obtain arbitrarily close tracking of position and velocity and transparency of the teleoperation system.

As a future work, it will be studied the validity of the proposed approach under the case of delays in the communication channel.

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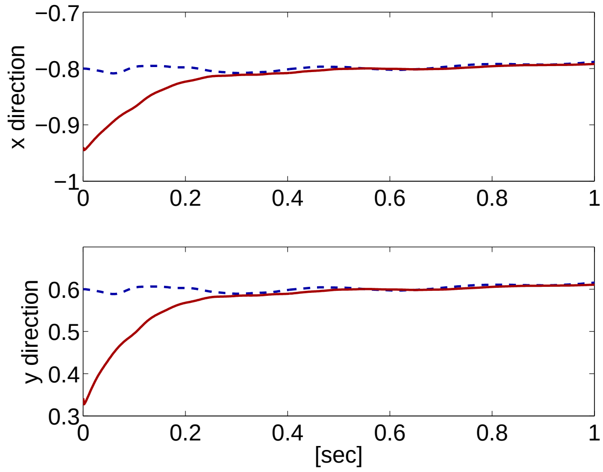

Figure 9. Zoom of Figure 8: $\boldsymbol{J}_{\varphi \mathrm{xs}}$ (- - -), $\hat{\boldsymbol{J}}_{\varphi \mathrm{xs}}$ (—).

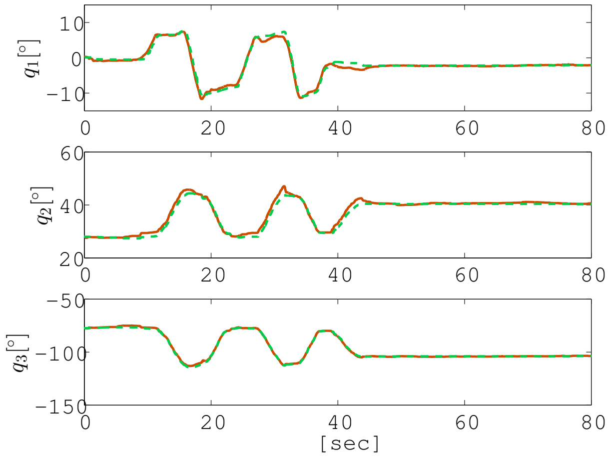

Figure 10. Position tracking in Cartesian coordinates: master (—), slave (—).

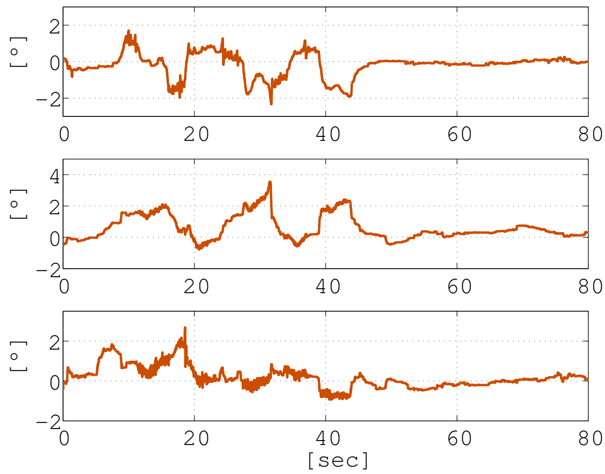

Figure 11. Position tracking error.

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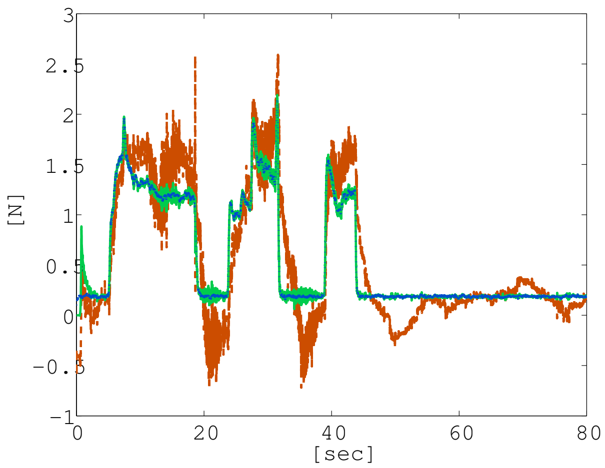

Figure 12. Force tracking and estimation: desired (—), estimated(—), virtual(- - -).

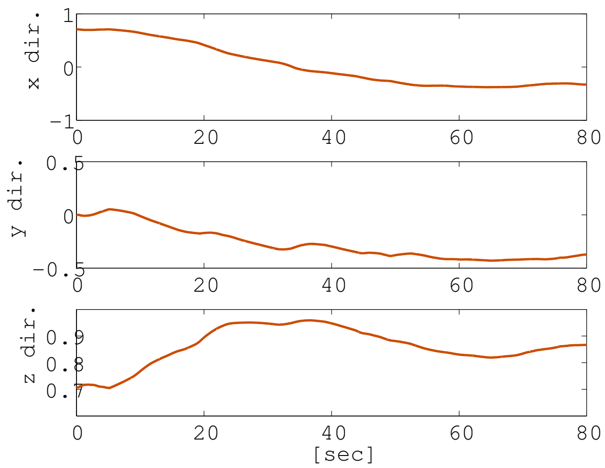

Figure 13. Estimation of the surface gradient: $\hat{\boldsymbol{J}}_{\varphi \mathrm{xs}}$.

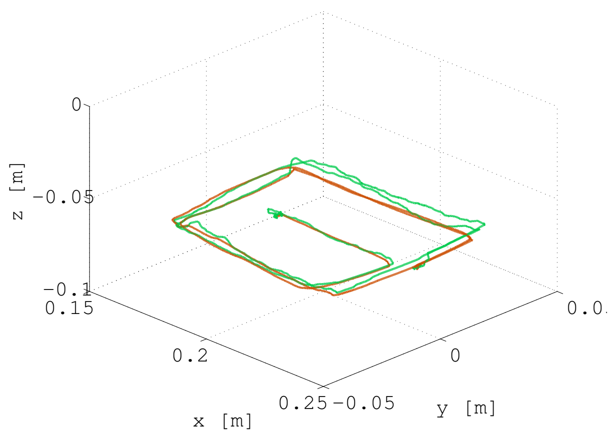

Figure 14. .

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Appendix A. Proof of Theorem 1

Theorem 1 is borrowed from Gutiérrez-Giles and Arteaga-Pérez (2016). The proof presented here belongs therefore to Gutiérrez-Giles and Arteaga-Pérez (2016) and it is included in this appendix only for the reviewers' convenience.

Theorem 1 states a local stability result, valid only in a region of interest $\mathcal{D}_s$, where Fact 1 holds.

Therefore, it must be shown that any signal of interest is bounded whenever $\boldsymbol{y}_{\mathrm{s}} \in \mathcal{D}_{\mathrm{s}}$ and that, with a proper choice of gains, $\boldsymbol{y}_{\mathrm{s}}$ will stay in $\mathcal{D}_{\mathrm{s}}$ for all time and will tend to an arbitrary small region around the origin. Consider the next four steps:

a) First, we show that whenever the state $\boldsymbol{y}_{\mathrm{s}} \in \mathcal{D}_{\mathrm{s}}$, then every signal of interest is also bounded. From (57) it is

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\sigma} = -\boldsymbol{K}_{\mathrm{vs}}^{-1}\bar{\boldsymbol{K}}_{\mathrm{is}}\boldsymbol{Q}_{\mathrm{s}}\boldsymbol{\sigma} + \boldsymbol{s}_{\mathrm{p}}\,, \tag{A1}$$

where $\boldsymbol{s}_{\mathrm{p}}$ is bounded in $\mathcal{D}_{\mathrm{s}}$, because due to Fact 1 $\boldsymbol{s}_{\mathrm{p}}$ and $\boldsymbol{s}_{\mathrm{F}}$ are orthogonal. For simplicity's sake consider $\boldsymbol{K}_{\mathrm{vs}} = k_{\mathrm{vs}}\boldsymbol{I}$ and $\bar{\boldsymbol{K}}_{\mathrm{is}} = \bar{k}_{\mathrm{is}}\boldsymbol{I}$. Then it can be shown that $\boldsymbol{\sigma}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\sigma}$ are bounded and $\|\boldsymbol{Q}_{\mathrm{s}}\boldsymbol{\sigma}\|$ and $\|\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\sigma}\|$ can be made arbitrarily small by setting $\bar{k}_{\mathrm{i}}$ large enough (see Gutiérrez-Giles and Arteaga-Pérez, 2014, for details). As a result, from (52) $\dot{\boldsymbol{e}}_{\mathrm{s}}$, $\boldsymbol{e}_{\mathrm{s}}$, and $\int_0^t \boldsymbol{e}_{\mathrm{s}}\,\mathrm{d}\vartheta$ are bounded. Since $\dot{\boldsymbol{q}}_{\mathrm{sd}}$ and $\boldsymbol{q}_{\mathrm{sd}}$ are bounded by assumption, then $\dot{\boldsymbol{q}}_{\mathrm{s}} = \boldsymbol{q}_{\mathrm{s2}}$ and $\boldsymbol{q}_{\mathrm{s}}$ must be bounded. Furthermore, $\hat{\boldsymbol{q}}_{\mathrm{s}}$, $\dot{\hat{\boldsymbol{q}}}_{\mathrm{s}}$, and $\hat{\boldsymbol{q}}_{\mathrm{s2}}$ are also bounded after (39) and because $\boldsymbol{x}_{\mathrm{o}}$ is bounded in $\mathcal{D}_{\mathrm{s}}$. From (60) it follows that $\hat{\boldsymbol{J}}_{\varphi\mathrm{xs}}$, $\hat{\boldsymbol{J}}_{\varphi\mathrm{s}}$, $\hat{\boldsymbol{Q}}_{\mathrm{xs}}$, and $\hat{\boldsymbol{Q}}_{\mathrm{s}}$ are bounded because $\bar{\boldsymbol{J}}_{\varphi\mathrm{xs}}$ is bounded in $\mathcal{D}_{\mathrm{s}}$. This implies after (23) that $\boldsymbol{\tau}_{\mathrm{s}}$ is bounded. Now, consider (Murray et al., 1994)

$$\begin{aligned}\boldsymbol{\lambda}_{\mathrm{s}} = {} & \left(\boldsymbol{J}_{\varphi\mathrm{s}}(\boldsymbol{q}_{\mathrm{s}})\boldsymbol{H}_{\mathrm{s}}^{-1}(\boldsymbol{q}_{\mathrm{s}})\boldsymbol{J}_{\varphi\mathrm{s}}^{\mathrm{T}}(\boldsymbol{q}_{\mathrm{s}})\right)^{-1}\left\{\boldsymbol{J}_{\varphi\mathrm{s}}(\boldsymbol{q}_{\mathrm{s}})\boldsymbol{H}_{\mathrm{s}}^{-1}(\boldsymbol{q}_{\mathrm{s}})\left(\boldsymbol{\tau}_{\mathrm{s}} - \boldsymbol{N}_{\mathrm{s}}(\boldsymbol{q}_{\mathrm{s}},\boldsymbol{q}_{\mathrm{s2}})\right)\right.\\ & \left. + \boldsymbol{J}_{\varphi\mathrm{s}}(\boldsymbol{q}_{\mathrm{s}},\boldsymbol{q}_{\mathrm{s2}})\boldsymbol{q}_{\mathrm{s2}}\right\}\,.\end{aligned} \tag{A2}$$

Since $\boldsymbol{H}_{\mathrm{s}}$ is bounded and positive definite, $\boldsymbol{\lambda}_{\mathrm{s}}$ is bounded, which in turn means that $\boldsymbol{z}_1$ in (9) is bounded too. By taking into account Assumptions 1 and 2, the partial derivatives $\partial\varphi_{\mathrm{s}}(\boldsymbol{q}_{\mathrm{s}})/\partial\boldsymbol{q}_{\mathrm{s}}, \partial^2\varphi_{\mathrm{s}}(\boldsymbol{q}_{\mathrm{s}})/\partial\boldsymbol{q}_{\mathrm{s}}^2, \ldots, \partial^{p+1}\varphi_{\mathrm{s}}(\boldsymbol{q}_{\mathrm{s}})/\partial\boldsymbol{q}_{\mathrm{s}}^{p+1}$ are bounded. Therefore, $\dot{\boldsymbol{J}}_{\varphi\mathrm{s}}(\boldsymbol{q}_{\mathrm{s}}) = (\partial\boldsymbol{J}_{\varphi\mathrm{s}}(\boldsymbol{q}_{\mathrm{s}})/\partial\boldsymbol{q}_{\mathrm{s}})\,\dot{\boldsymbol{q}}_{\mathrm{s}}$ must be bounded as well as $\boldsymbol{f}_{\mathrm{s}}(t)$ in (45) and $\dot{\boldsymbol{q}}_{\mathrm{r}}$ in (56). From (44) one can conclude that $\tilde{\boldsymbol{z}}_1$ is bounded and, as a consequence, $\hat{\boldsymbol{z}}_1$ and $\hat{\lambda}_{\mathrm{s}}$ in (18), and $\Delta\bar{\lambda}_{\mathrm{s}}$ in (24) must be bounded. After (19) and (61), $\dot{\hat{\boldsymbol{J}}}_{\varphi\mathrm{s}}$ and $\dot{\hat{\boldsymbol{J}}}_{\varphi\mathrm{s}}$ are bounded as well.

Taking into account (A2) and the slave manipulator model (7)–(9), one can write the joint acceleration as a function of only $(\boldsymbol{q}_{\mathrm{s}}, \boldsymbol{q}_{\mathrm{s2}}, \boldsymbol{\tau}_{\mathrm{s}})$, *i.e.*,

$$\dot{\boldsymbol{q}}_{\mathrm{s2}} = \ddot{\boldsymbol{q}}_{\mathrm{s}} = \boldsymbol{f}_{\mathrm{q}}(\boldsymbol{q}_{\mathrm{s}}, \dot{\boldsymbol{q}}_{\mathrm{s}}, \boldsymbol{\tau}_{\mathrm{s}})\,, \tag{A3}$$

which clearly shows that $\dot{\boldsymbol{q}}_{\mathrm{s2}}$ is bounded. Since $\ddot{\boldsymbol{q}}_{\mathrm{sd}}$ and $\ddot{\hat{\boldsymbol{q}}}_{\mathrm{s}}$ are bounded, $\ddot{\boldsymbol{e}}_{\mathrm{s}}$, $\ddot{\hat{\boldsymbol{q}}}_{\mathrm{s}}$ and after (39) $\dot{\hat{\boldsymbol{q}}}_{\mathrm{s2}}$ must be bounded as well. Now, by similar arguments, $\boldsymbol{\tau}_{\mathrm{s}}$ in (23) can be written as a function of bounded variables, *i.e.*,

$$\boldsymbol{\tau}_{\mathrm{s}} = \boldsymbol{f}_{\tau}\left(\dot{\boldsymbol{q}}_{\mathrm{sd}}, \int_0^t \boldsymbol{e}_{\mathrm{s}}\,\mathrm{d}\vartheta, \boldsymbol{e}_{\mathrm{s}}, \tilde{\boldsymbol{q}}_{\mathrm{s}}, \dot{\hat{\boldsymbol{q}}}_{\mathrm{s}}, \hat{\boldsymbol{J}}_{\varphi\mathrm{s}}, \lambda_{\mathrm{sd}}, \Delta\bar{F}_{\mathrm{s}}\right)\,. \tag{A4}$$

Therefore, its time derivative must be a function of the form

$$\dot{\boldsymbol{\tau}}_{\mathrm{s}} = \dot{\boldsymbol{f}}_{\tau}\left(\dot{\boldsymbol{q}}_{\mathrm{sd}}, \ddot{\boldsymbol{q}}_{\mathrm{sd}}, \int_0^t \boldsymbol{e}_{\mathrm{s}}\,\mathrm{d}\vartheta, \boldsymbol{e}_{\mathrm{s}}, \dot{\boldsymbol{e}}_{\mathrm{s}}, \tilde{\boldsymbol{q}}_{\mathrm{s}}, \dot{\hat{\boldsymbol{q}}}_{\mathrm{s}}, \ddot{\hat{\boldsymbol{q}}}_{\mathrm{s}}, \hat{\boldsymbol{J}}_{\varphi\mathrm{s}}, \dot{\hat{\boldsymbol{J}}}_{\varphi\mathrm{s}}, \lambda_{\mathrm{sd}}, \dot{\lambda}_{\mathrm{sd}}, \Delta\bar{F}_{\mathrm{s}}, \Delta\bar{\lambda}_{\mathrm{s}}\right)\,, \tag{A5}$$

which is bounded since it depends on variables we have already proven to be bounded. The time derivative of $\dot{\boldsymbol{q}}_{\mathrm{r}}$ in (56) is given by

$$\begin{aligned}\ddot{\boldsymbol{q}}_{\mathrm{r}} = \ddot{\boldsymbol{q}}_{\mathrm{sd}} - \boldsymbol{\Lambda}\dot{\boldsymbol{e}}_{\mathrm{s}} - \boldsymbol{K}_{\mathrm{vs}}^{-1}\bar{\boldsymbol{K}}_{\mathrm{is}}\dot{\boldsymbol{Q}}_{\mathrm{s}}\boldsymbol{\sigma} - \boldsymbol{K}_{\mathrm{vs}}^{-1}\bar{\boldsymbol{K}}_{\mathrm{is}}\boldsymbol{Q}_{\mathrm{s}}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\sigma} + \frac{1}{2}\boldsymbol{K}_{\mathrm{vs}}^{-1}\dot{\boldsymbol{J}}_{\varphi s}^{\mathrm{T}}\Delta\lambda_{\mathrm{s}} \\ + \frac{1}{2}\boldsymbol{K}_{\mathrm{vs}}^{-1}\boldsymbol{J}_{\varphi s}^{\mathrm{T}}\frac{\mathrm{d}}{\mathrm{d}t}(\Delta\lambda_{\mathrm{s}}) + \frac{1}{2}k_{\mathrm{Fis}}\boldsymbol{K}_{\mathrm{vs}}^{-1}\dot{\boldsymbol{J}}_{\varphi s}^{+}\Delta\bar{F}_{\mathrm{s}} + \frac{1}{2}k_{\mathrm{Fis}}\boldsymbol{K}_{\mathrm{vs}}^{-1}\boldsymbol{J}_{\varphi s}^{+}\Delta\bar{\lambda}_{\mathrm{s}}\ ,\end{aligned} \tag{A6}$$

which again turns out to be bounded since from (A2), $\frac{\mathrm{d}}{\mathrm{d}t}(\Delta\lambda_{\mathrm{s}})$ is a function only of $(\boldsymbol{q}_{\mathrm{s}}, \dot{\boldsymbol{q}}_{\mathrm{s}}, \ddot{\boldsymbol{q}}_{\mathrm{s}}, \boldsymbol{\tau}_{\mathrm{s}}, \dot{\boldsymbol{\tau}}_{\mathrm{s}})$. Then, there must exist a positive constant c_{a} such that $\boldsymbol{y}_{\mathrm{a}}$ in (59) fulfils $\|\boldsymbol{y}_{\mathrm{a}}\| \leq c_{\mathrm{a}}$, whenever $\boldsymbol{y}_{\mathrm{s}} \in \mathcal{D}_{\mathrm{s}}$. As a direct consequence, $\dot{\boldsymbol{s}}$ in (58) is bounded. By differentiating (57) one obtains

$$\begin{aligned}\dot{\boldsymbol{s}} = \ddot{\boldsymbol{\sigma}} + \boldsymbol{K}_{\mathrm{vs}}^{-1}\bar{\boldsymbol{K}}_{\mathrm{is}}\dot{\boldsymbol{Q}}_{\mathrm{s}}\boldsymbol{\sigma} + \boldsymbol{K}_{\mathrm{vs}}^{-1}\bar{\boldsymbol{K}}_{\mathrm{is}}\boldsymbol{Q}_{\mathrm{s}}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\sigma} - \frac{1}{2}\boldsymbol{K}_{\mathrm{vs}}^{-1}\dot{\boldsymbol{J}}_{\varphi s}^{\mathrm{T}}\Delta\lambda_{\mathrm{s}} - \frac{1}{2}\boldsymbol{K}_{\mathrm{vs}}^{-1}\boldsymbol{J}_{\varphi s}^{\mathrm{T}}\frac{\mathrm{d}}{\mathrm{d}t}(\Delta\lambda_{\mathrm{s}}) \\ - \frac{1}{2}k_{\mathrm{Fis}}\boldsymbol{K}_{\mathrm{vs}}^{-1}\dot{\boldsymbol{J}}_{\varphi s}^{+}\Delta\bar{F}_{\mathrm{s}} - \frac{1}{2}k_{\mathrm{Fis}}\boldsymbol{K}_{\mathrm{vs}}^{-1}\boldsymbol{J}_{\varphi s}^{+}\Delta\bar{\lambda}_{\mathrm{s}}\,,\end{aligned} \tag{A7}$$

so that $\ddot{\boldsymbol{\sigma}}$ must be bounded.

At this point, an iterative argument is carried out. First, by computing the time derivative of (A3) it is

$$\boldsymbol{q}_{\mathrm{s}}^{(3)} = \dot{\boldsymbol{f}}_{\mathrm{q}}(\boldsymbol{q}_{\mathrm{s}}, \dot{\boldsymbol{q}}_{\mathrm{s}}, \ddot{\boldsymbol{q}}_{\mathrm{s}}, \boldsymbol{\tau}_{\mathrm{s}}, \dot{\boldsymbol{\tau}}_{\mathrm{s}})\,, \tag{A8}$$

which shows that $\boldsymbol{q}_{\mathrm{s}}^{(3)}$, $\boldsymbol{e}_{\mathrm{s}}^{(3)}$ and $\hat{\boldsymbol{q}}_{\mathrm{s}}^{(3)}$ are bounded. Combining (13) and (14), it can be written

$$\hat{\boldsymbol{z}}_1 = \boldsymbol{f}_{\hat{\mathrm{z}}_1}(\boldsymbol{q}_{\mathrm{s}}, \ddot{\hat{\boldsymbol{q}}}_{\mathrm{s}}, \tilde{\boldsymbol{q}}_{\mathrm{s}}, \dot{\tilde{\boldsymbol{q}}}_{\mathrm{s}}, \hat{\boldsymbol{q}}_{\mathrm{s}2}, \boldsymbol{\tau}_{\mathrm{s}})\,. \tag{A9}$$

This implies that

$$\dot{\hat{\boldsymbol{z}}}_1 = \dot{\boldsymbol{f}}_{\hat{\mathrm{z}}_1}\left(\boldsymbol{q}_{\mathrm{s}}, \dot{\boldsymbol{q}}_{\mathrm{s}}, \ddot{\hat{\boldsymbol{q}}}_{\mathrm{s}}, \hat{\boldsymbol{q}}_{\mathrm{s}}^{(3)}, \tilde{\boldsymbol{q}}_{\mathrm{s}}, \dot{\tilde{\boldsymbol{q}}}_{\mathrm{s}}, \ddot{\tilde{\boldsymbol{q}}}_{\mathrm{s}}, \hat{\boldsymbol{q}}_{\mathrm{s}2}, \dot{\hat{\boldsymbol{q}}}_{\mathrm{s}2}, \boldsymbol{\tau}_{\mathrm{s}}, \dot{\boldsymbol{\tau}}_{\mathrm{s}}\right) \tag{A10}$$

is bounded and so are $\dot{\hat{\lambda}}_{\mathrm{s}}$ and $\frac{\mathrm{d}}{\mathrm{d}t}(\Delta\bar{\lambda}_{\mathrm{s}})$ as a consequence. From (19) it is

$$\dot{\hat{\boldsymbol{J}}}_{\varphi\mathrm{xs}} = \dot{\boldsymbol{f}}_{\hat{\mathrm{J}}_{\varphi\mathrm{xs}}}(\boldsymbol{q}_{\mathrm{s}}, \hat{\boldsymbol{z}}_1)\,, \tag{A11}$$

where (18) has been taken into account. Then, after (21) it means

$$\dot{\hat{\boldsymbol{J}}}_{\varphi\mathrm{s}} = \dot{\boldsymbol{f}}_{\hat{\mathrm{J}}_{\varphi\mathrm{s}}}(\boldsymbol{q}_{\mathrm{s}}, \dot{\boldsymbol{q}}_{\mathrm{s}}, \hat{\boldsymbol{z}}_1) \tag{A12}$$

$$\ddot{\hat{\boldsymbol{J}}}_{\varphi\mathrm{s}} = \ddot{\boldsymbol{f}}_{\hat{\mathrm{J}}_{\varphi\mathrm{s}}}(\boldsymbol{q}_{\mathrm{s}}, \dot{\boldsymbol{q}}_{\mathrm{s}}, \ddot{\boldsymbol{q}}_{\mathrm{s}}, \hat{\boldsymbol{z}}_1, \dot{\hat{\boldsymbol{z}}}_1)\,, \tag{A13}$$

which implies that $\ddot{\boldsymbol{Q}}_{\mathrm{s}}$ is bounded. On the other hand

$$\ddot{\boldsymbol{J}}_{\varphi\mathrm{s}} = \boldsymbol{f}_{\mathrm{J}_{\varphi\mathrm{s}}}(\boldsymbol{q}_{\mathrm{s}}, \dot{\boldsymbol{q}}_{\mathrm{s}}, \ddot{\boldsymbol{q}}_{\mathrm{s}}) \tag{A14}$$

must be bounded from Assumption 2 and because $\dot{\boldsymbol{q}}_{\mathrm{s}}$ and $\dot{\boldsymbol{q}}_{\mathrm{s}2}$ are bounded, which along with (A13) means that $\ddot{\hat{\boldsymbol{J}}}_{\varphi\mathrm{s}}$ is bounded. Now, from (A5) it is

$$\ddot{\boldsymbol{\tau}}_{\mathrm{s}} = \ddot{\boldsymbol{f}}_{\tau}\left(\dot{\boldsymbol{q}}_{\mathrm{sd}},\ldots,\boldsymbol{q}_{\mathrm{sd}}^{(3)},\int_0^t \boldsymbol{e}_{\mathrm{s}}\,\mathrm{d}\vartheta,\boldsymbol{e}_{\mathrm{s}},\ldots,\ddot{\boldsymbol{e}}_{\mathrm{s}},\tilde{\boldsymbol{q}}_{\mathrm{s}},\ldots,\tilde{\boldsymbol{q}}_{\mathrm{s}}^{(3)},\right.$$
$$\left.\hat{\boldsymbol{J}}_{\varphi\mathrm{s}},\ldots,\ddot{\hat{\boldsymbol{J}}}_{\varphi\mathrm{s}},\lambda_{\mathrm{sd}},\ldots,\ddot{\lambda}_{\mathrm{sd}},\Delta\bar{F}_{\mathrm{s}},\Delta\bar{\lambda}_{\mathrm{s}},\frac{\mathrm{d}}{\mathrm{d}t}(\Delta\bar{\lambda}_{\mathrm{s}})\right)\,, \quad \text{(A15)}$$

which is bounded from the same arguments as those in the previous discussion. By the definition of $\boldsymbol{f}_{\mathrm{s}}$ in (45) it can be written

$$\boldsymbol{f}_{\mathrm{s}} = \boldsymbol{f}_{\mathrm{s}}(\boldsymbol{q}_{\mathrm{s}},\dot{\boldsymbol{q}}_{\mathrm{s}},\dot{\boldsymbol{q}}_{\mathrm{sd}},\tilde{\boldsymbol{q}}_{\mathrm{s}},\dot{\tilde{\boldsymbol{q}}}_{\mathrm{s}}) \quad \text{(A16)}$$
$$\dot{\boldsymbol{f}}_{\mathrm{s}} = \dot{\boldsymbol{f}}_{\mathrm{s}}(\boldsymbol{q}_{\mathrm{s}},\dot{\boldsymbol{q}}_{\mathrm{s}},\ddot{\boldsymbol{q}}_{\mathrm{s}},\dot{\boldsymbol{q}}_{\mathrm{sd}},\ddot{\boldsymbol{q}}_{\mathrm{sd}},\tilde{\boldsymbol{q}}_{\mathrm{s}},\dot{\tilde{\boldsymbol{q}}}_{\mathrm{s}},\ddot{\tilde{\boldsymbol{q}}}_{\mathrm{s}})\,. \quad \text{(A17)}$$

As a result, from (44) one can conclude that $\dot{\tilde{\boldsymbol{z}}}_1$ is bounded, and so is $\dot{\hat{\boldsymbol{z}}}_1$. Following this procedure iteratively, it is obtained

$$\boldsymbol{q}_{\mathrm{s}}^{(p+1)} = \boldsymbol{f}_{\mathrm{q}}^{(p-1)}\left(\boldsymbol{q}_{\mathrm{s}},\ldots,\boldsymbol{q}_{\mathrm{s}}^{(p)},\boldsymbol{\tau}_{\mathrm{s}},\ldots,\boldsymbol{\tau}_{\mathrm{s}}^{(p-1)}\right)\,, \quad \text{(A18)}$$

which means that $\boldsymbol{q}_{\mathrm{s}}^{(p+1)}$ and $\hat{\boldsymbol{q}}_{\mathrm{s}}^{(p+1)}$ (and all their previous derivatives) are bounded. From (A10) it follows

$$\hat{\boldsymbol{z}}_1^{(p-1)} = \boldsymbol{f}_{\hat{\mathrm{z}}_1}^{(p-1)}\left(\boldsymbol{q}_{\mathrm{s}},\ldots,\boldsymbol{q}_{\mathrm{s}}^{(p-1)},\hat{\boldsymbol{q}}_{\mathrm{s}},\ldots,\hat{\boldsymbol{q}}_{\mathrm{s}}^{(p+1)},\tilde{\boldsymbol{q}}_{\mathrm{s}},\ldots,\tilde{\boldsymbol{q}}_{\mathrm{s}}^{(p)},\right.$$
$$\left.\dot{\hat{\boldsymbol{q}}}_{\mathrm{s2}},\ldots\hat{\boldsymbol{q}}_{\mathrm{s2}}^{(p-1)},\boldsymbol{\tau}_{\mathrm{s}},\ldots,\boldsymbol{\tau}_{\mathrm{s}}^{(p-1)}\right)\,, \quad \text{(A19)}$$

which must be bounded as well as all its previous time derivatives. It also implies that $\ddot{\hat{\lambda}}_{\mathrm{s}},\ldots,\hat{\lambda}_{\mathrm{s}}^{(p-1)}$ and $\mathrm{d}^2(\Delta\bar{\lambda}_{\mathrm{s}})/\mathrm{d}t^2,\ldots,\mathrm{d}^{p-1}(\Delta\bar{\lambda}_{\mathrm{s}})/\mathrm{d}t^{p-1}$ are bounded. In the same manner, from (A13) it is

$$\hat{\boldsymbol{J}}_{\varphi\mathrm{s}}^{(p)} = \boldsymbol{f}_{\hat{\mathrm{J}}_{\varphi\mathrm{s}}}^{(p)}\left(\boldsymbol{q}_{\mathrm{s}},\ldots,\boldsymbol{q}_{\mathrm{s}}^{(p)},\hat{\boldsymbol{z}}_1,\ldots,\hat{\boldsymbol{z}}_1^{(p-1)}\right)\,. \quad \text{(A20)}$$

Also, from (A15) it is obtained

$$\boldsymbol{\tau}_{\mathrm{s}}^{(p)} = \boldsymbol{f}_{\tau}^{(p)}\left(\dot{\boldsymbol{q}}_{\mathrm{sd}},\ldots,\boldsymbol{q}_{\mathrm{sd}}^{(p+1)},\int_0^t \boldsymbol{e}_{\mathrm{s}}\,\mathrm{d}\vartheta,\boldsymbol{e}_{\mathrm{s}},\ldots,\boldsymbol{e}_{\mathrm{s}}^{(p)},\tilde{\boldsymbol{q}}_{\mathrm{s}},\ldots,\tilde{\boldsymbol{q}}_{\mathrm{s}}^{(p+1)},\right.$$
$$\left.\hat{\boldsymbol{J}}_{\varphi\mathrm{s}},\ldots,\hat{\boldsymbol{J}}_{\varphi\mathrm{s}}^{(p)},\lambda_{\mathrm{sd}},\ldots,\lambda_{\mathrm{sd}}^{(p)},\Delta\bar{F}_{\mathrm{s}},\Delta\bar{\lambda}_{\mathrm{s}},\ldots,\mathrm{d}^{(p-1)}(\Delta\bar{\lambda}_{\mathrm{s}})/\mathrm{d}t^{(p-1)}\right)\,,$$

that is bounded, since it is a function of bounded signals. On the other hand, from (A17) it can be written

$$\boldsymbol{f}_{\mathrm{s}}^{(p)} = \boldsymbol{f}_{\mathrm{s}}^{(p)}\left(\boldsymbol{q}_{\mathrm{s}},\ldots,\boldsymbol{q}_{\mathrm{s}}^{(p+1)},\dot{\boldsymbol{q}}_{\mathrm{sd}},\ldots,\boldsymbol{q}_{\mathrm{sd}}^{(p+1)},\tilde{\boldsymbol{q}}_{\mathrm{s}},\ldots,\tilde{\boldsymbol{q}}_{\mathrm{s}}^{(p+1)}\right)\,, \quad \text{(A21)}$$

which is bounded with all its previous derivatives bounded too. From (44) it is computed

$$\tilde{\boldsymbol{q}}_{\mathrm{s}}^{(p+1)} + \boldsymbol{\lambda}_{p+1}\tilde{\boldsymbol{q}}_{\mathrm{s}}^{(p)} + \boldsymbol{\lambda}_{p}\tilde{\boldsymbol{q}}_{\mathrm{s}}^{(p-1)} = \tilde{\boldsymbol{z}}_1^{(p-1)} + \boldsymbol{f}_{\mathrm{s}}^{(p-1)}\,, \quad \text{(A22)}$$

which implies that $(\tilde{\boldsymbol{z}}_1, \ldots, \tilde{\boldsymbol{z}}_1^{(p-1)})$ are bounded too. Also, from (9) and (A2) it can be seen that

$$\boldsymbol{z}_1 = \boldsymbol{f}_{\mathrm{z}_1}(\boldsymbol{q}_{\mathrm{s}}, \dot{\boldsymbol{q}}_{\mathrm{s}}, \boldsymbol{\tau}_{\mathrm{s}}) \,, \tag{A23}$$

from where it can be stated that all its time derivatives up to

$$\boldsymbol{z}_1^{(p)} = \boldsymbol{f}_{\mathrm{z}_1}^{(p)} \left(\boldsymbol{q}_{\mathrm{s}}, \ldots, \boldsymbol{q}_{\mathrm{s}}^{(p+1)}, \boldsymbol{\tau}_{\mathrm{s}}, \ldots, \boldsymbol{\tau}_{\mathrm{s}}^{(p)} \right) \tag{A24}$$

must be bounded as well. From (10)–(12) it can be concluded that $(\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_p, \dot{\boldsymbol{z}}_1, \dot{\boldsymbol{z}}_2, \ldots, \dot{\boldsymbol{z}}_p)$ and $\boldsymbol{r}^{(p)}$ must be bounded, since $\boldsymbol{z}_p = \boldsymbol{z}_1^{(p)}$. Furthermore, from (17), one can see that $\dot{\hat{\boldsymbol{z}}}_p$ is bounded as well. Also, since $(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_1^{(p-1)})$ are bounded, one can easily show that the estimated variables $(\hat{\boldsymbol{z}}_2, \ldots, \hat{\boldsymbol{z}}_p, \dot{\hat{\boldsymbol{z}}}_2, \ldots, \dot{\hat{\boldsymbol{z}}}_{p-1})$ must be bounded. Moreover, all the related errors must be bounded as well.

b) The second step of the proof is completely analogous to that given in Gutiérrez-Giles and Arteaga-Pérez (2014) from which only the main points are recalled. Let

$$V_{\mathrm{a}} = \boldsymbol{x}_{\mathrm{o}}^{\mathrm{T}} \boldsymbol{P}_{\mathrm{o}} \boldsymbol{x}_{\mathrm{o}} \,, \tag{A25}$$

with $\boldsymbol{x}_{\mathrm{o}}$ defined in (48) and $\boldsymbol{P}_{\mathrm{o}} = \boldsymbol{P}_{\mathrm{o}}^{\mathrm{T}} > \boldsymbol{O}$ given as the solution of

$$\boldsymbol{A}^{\mathrm{T}} \boldsymbol{P}_{\mathrm{o}} + \boldsymbol{P}_{\mathrm{o}} \boldsymbol{A} = -\boldsymbol{Q}_{\mathrm{o}} \,, \tag{A26}$$

where $\boldsymbol{Q}_{\mathrm{o}}$ is a positive definite matrix and $\boldsymbol{A}$ is given by (49). Whenever $\boldsymbol{y}_{\mathrm{s}} \in \mathcal{D}_{\mathrm{s}}$, $\boldsymbol{r}_{\mathrm{f}}$ in (47) is bounded and there must exist a constant, say $r_{\max}$, such that $\sup_{t_0 \le \vartheta \le t} \|\boldsymbol{r}_{\mathrm{f}}(\vartheta)\| \le r_{\max}$. The time derivative of V_{a} fulfils

$$\dot{V}_{\mathrm{a}} \le -\|\boldsymbol{x}_{\mathrm{o}}\| \left(\lambda_{\min}(\boldsymbol{Q}_{\mathrm{o}}) \|\boldsymbol{x}_{\mathrm{o}}\| - 2\lambda_{\max}(\boldsymbol{P}_{\mathrm{o}}) \|\boldsymbol{B}\| r_{\max} \right) \,. \tag{A27}$$

Then, it follows

$$\dot{V}_{\mathrm{a}} \le 0 \quad \text{if} \quad \|\boldsymbol{x}_{\mathrm{o}}(t)\| \ge \frac{2\lambda_{\max}(\boldsymbol{P}_{\mathrm{o}})}{\lambda_{\min}(\boldsymbol{Q}_{\mathrm{o}})} \|\boldsymbol{B}\| r_{\max} \,, \tag{A28}$$

where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the minimum and the maximum eigenvalue of their arguments, respectively. Also, since the system (47) is linear, by properly choosing the eigenvalues of $\boldsymbol{A}$, there must exist an ultimate bound for $\boldsymbol{x}_{\mathrm{o}}$ given by (Arteaga-Pérez and Gutiérrez-Giles, 2014)

$$\|\boldsymbol{x}_{\mathrm{o}}(t)\| \le \frac{n(p+1)}{|\lambda_{\max}(\boldsymbol{A})|} \|\boldsymbol{B}\| r_{\max}, \quad \text{as} \quad t \to \infty \,. \tag{A29}$$

Since $n(p+1)$ is fixed and $|\lambda_{\max}(\boldsymbol{A})|$ can be chosen arbitrarily large, the ultimate bound of $\boldsymbol{x}_{\mathrm{o}}$ can be made arbitrarily small. Besides, it can be proved that the ultimate bound of $\boldsymbol{x}_{\mathrm{o}}$ also fulfils (Khalil, 2002)

$$\|\boldsymbol{x}_{\mathrm{o}}(t)\| \le \frac{2\lambda_{\max}(\boldsymbol{P}_{\mathrm{o}})}{\lambda_{\min}(\boldsymbol{Q}_{\mathrm{o}})} \sqrt{\frac{\lambda_{\max}(\boldsymbol{P}_{\mathrm{o}})}{\lambda_{\min}(\boldsymbol{P}_{\mathrm{o}})}} \|\boldsymbol{B}\| r_{\max} \,, \tag{A30}$$

which means (*c.f.* (A29)), that the term $\lambda_{\max}(\boldsymbol{P}_{\rm o})/\lambda_{\min}(\boldsymbol{Q}_{\rm o})$ can be made arbitrarily small. Notice that the ultimate bound of $\|\boldsymbol{x}_{\rm o}\|$ in (A29) can be made arbitrarily small independently of the norm of $\boldsymbol{y}_{\rm s}$, *i.e.,* it depends only on the choice of the eigenvalues of $\boldsymbol{A}$ in (49).

c) Till now it has been shown that whenever $\boldsymbol{y}_{\rm s} \in \mathcal{D}_{\rm s}$, every signal of interest is bounded and furthermore, the observation errors can be made arbitrarily small independently of the rest of the state error. The next step is to show that whenever $\boldsymbol{y}_{\rm s}(t_0)$ is small enough, then it can be enforced for $\boldsymbol{y}_{\rm s}$ to remain in $\mathcal{D}_{\rm s}$ and that actually $\|\boldsymbol{y}_{\rm s}\|$ can be made arbitrarily small, *i.e.,* $\boldsymbol{y}_{\rm s}$ is ultimately bounded with ultimate bound arbitrarily small (see Gutiérrez-Giles and Arteaga-Pérez, 2014, Figure A1).

Let

$$V_{\rm s} = \frac{1}{2}\boldsymbol{s}^{\rm T}\boldsymbol{H}_{\rm s}(\boldsymbol{q})\boldsymbol{s} + \frac{1}{4}\frac{k_{\rm Fis}}{k_{\rm vs}}(\Delta\bar{F}_{\rm s})^2 \tag{A31}$$

be a positive function of $\boldsymbol{s}$ and $\Delta\bar{F}_{\rm s}$. From Properties 2–4, it can be shown that the time derivative of (A31) along (25) and (58) is given by

$$\begin{aligned}\dot{V}_{\rm s} =& -\boldsymbol{s}^{\rm T}\boldsymbol{K}_{\rm Dvs}\boldsymbol{s} + \boldsymbol{s}^{\rm T}\boldsymbol{y}_{\rm a} - \frac{1}{4}k_{\rm vs}^{-1}(\Delta\lambda_{\rm s})^2\boldsymbol{J}_{\varphi\rm s}\boldsymbol{J}_{\varphi\rm s}^{\rm T} - \frac{1}{2}\frac{k_{\rm Fis}}{k_{\rm vs}}\Delta\bar{F}_{\rm s}(\boldsymbol{J}_{\varphi\rm s}^{+})^{\rm T}\boldsymbol{H}_{\rm s}(\boldsymbol{q}_{\rm s})\tilde{\boldsymbol{z}}_1\\ &-\frac{1}{4}\frac{k_{\rm Fis}^2}{k_{\rm vs}}(\Delta\bar{F}_{\rm s})^2(\boldsymbol{J}_{\varphi\rm s}\boldsymbol{J}_{\varphi\rm s}^{\rm T})^{-1}\\ \le& -k_{\rm vs}\|\boldsymbol{s}\|^2 + c_{\rm a}\|\boldsymbol{s}\| + \frac{1}{2}\frac{k_{\rm Fis}\lambda_{\rm H}c_{\varphi}^{+}}{k_{\rm vs}}|\Delta\bar{F}_{\rm s}|\|\tilde{\boldsymbol{z}}_1\| - \frac{1}{4}\frac{k_{\rm Fis}^2c_{\varphi}^{-}}{k_{\rm vs}}|\Delta\bar{F}_{\rm s}|^2\,,\end{aligned} \tag{A32}$$

where $c_{\varphi}^{+} \triangleq \|\boldsymbol{J}_{\varphi\rm s}^{+}\|_{\max}$ and $c_{\varphi}^{-} \triangleq \inf_{\boldsymbol{q}_{\rm s}\in\mathbb{R}^n}\left\{\boldsymbol{J}_{\varphi\rm s}(\boldsymbol{q}_{\rm s})\boldsymbol{J}_{\varphi\rm s}^{\rm T}(\boldsymbol{q}_{\rm s})\right\}^{-1}$. Note that, since $\boldsymbol{J}_{\varphi\rm s}(\boldsymbol{q}_{\rm s})$ is full rank for every $\boldsymbol{q}_{\rm s} \in \mathbb{R}^n$ and the Jacobian of the manipulator is non–singular and upper bounded, it is $0 < c_{\varphi}^{\pm} < \infty$. On the other hand, define now

$$V_{\varphi\rm xs} = \frac{1}{2}\tilde{\boldsymbol{J}}_{\varphi\rm xs}\tilde{\boldsymbol{J}}_{\varphi\rm xs}^{\rm T}\,. \tag{A33}$$

Its time derivative is given by

$$\dot{V}_{\varphi\rm xs} = \tilde{\boldsymbol{J}}_{\varphi\rm xs}\dot{\tilde{\boldsymbol{J}}}_{\varphi\rm xs}^{\rm T} = -\tilde{\boldsymbol{J}}_{\varphi\rm xs}\dot{\hat{\boldsymbol{J}}}_{\varphi\rm xs}^{\rm T} + \tilde{\boldsymbol{J}}_{\varphi\rm xs}\dot{\boldsymbol{J}}_{\varphi\rm xs}^{\rm T}\,. \tag{A34}$$

After (9) and (19), and since $\tilde{\boldsymbol{z}}_1 = \boldsymbol{z}_1 - \hat{\boldsymbol{z}}_1$, one obtains

$$\dot{V}_{\varphi\rm xs} = -\frac{\gamma}{\hat{\lambda}_{\rm s}+\epsilon}\tilde{\boldsymbol{J}}_{\varphi\rm xs}\hat{\boldsymbol{Q}}_{\rm xs}\left(\boldsymbol{J}_{\varphi\rm xs}^{\rm T}\lambda_{\rm s} - \boldsymbol{J}_{\rm s}^{-\rm T}(\boldsymbol{q}_{\rm s})\boldsymbol{H}_{\rm s}(\boldsymbol{q}_{\rm s})\tilde{\boldsymbol{z}}_1\right) + \tilde{\boldsymbol{J}}_{\varphi\rm xs}\dot{\boldsymbol{J}}_{\varphi\rm xs}^{\rm T}\,. \tag{A35}$$

Hence, because $\hat{\boldsymbol{Q}}_{\rm xs}\hat{\boldsymbol{J}}_{\varphi\rm xs}^{\rm T} = \boldsymbol{O}$ and $\hat{\boldsymbol{Q}}_{\rm xs}\hat{\boldsymbol{Q}}_{\rm xs} = \hat{\boldsymbol{Q}}_{\rm xs} = \hat{\boldsymbol{Q}}_{\rm xs}^{\rm T}$, it is

$$\dot{V}_{\varphi\rm xs} = -\frac{\gamma}{\hat{\lambda}_{\rm s}+\epsilon}\left(\tilde{\boldsymbol{J}}_{\varphi\rm xs}\hat{\boldsymbol{Q}}_{\rm xs}^{\rm T}\hat{\boldsymbol{Q}}_{\rm xs}\tilde{\boldsymbol{J}}_{\varphi\rm xs}^{\rm T}\lambda_{\rm s} - \tilde{\boldsymbol{J}}_{\varphi\rm xs}\hat{\boldsymbol{Q}}_{\rm xs}\boldsymbol{J}_{\rm s}^{-\rm T}(\boldsymbol{q}_{\rm s})\boldsymbol{H}_{\rm s}(\boldsymbol{q}_{\rm s})\tilde{\boldsymbol{z}}_1\right) + \tilde{\boldsymbol{J}}_{\varphi\rm xs}\dot{\boldsymbol{J}}_{\varphi\rm xs}^{\rm T}\,. \tag{A36}$$

Since $\boldsymbol{q}_{\rm s}$ and $\dot{\boldsymbol{q}}_{\rm s}$ are bounded in $\mathcal{D}_{\rm s}$ and the surface is assumed to be smooth, there must exist a positive constant, say $v_{\rm x}$, such that $\|\dot{\boldsymbol{J}}_{\varphi\rm xs}\| \le v_{\rm x} < \infty$. Also, consider a closed subset of the workspace of the manipulator centred at $\{\boldsymbol{x}_{\rm s} \in \mathbb{R}^n | \tilde{\boldsymbol{J}}_{\varphi\rm xs}(\boldsymbol{x}_{\rm s}) = \boldsymbol{0}\}$ and defined by $\mathcal{S} = \{\boldsymbol{x}_{\rm s} \in \mathbb{R}^n | \, \|\tilde{\boldsymbol{J}}_{\varphi\rm xs}(\boldsymbol{x}_{\rm s})\| \le \sqrt{2}\}$, *i.e.,* the region of the workspace where the angle, say α, between the normal to the surface $\boldsymbol{J}_{\varphi\rm xs}$ and its estimate $\hat{\boldsymbol{J}}_{\varphi\rm xs}$ is at most 90°, or equivalently

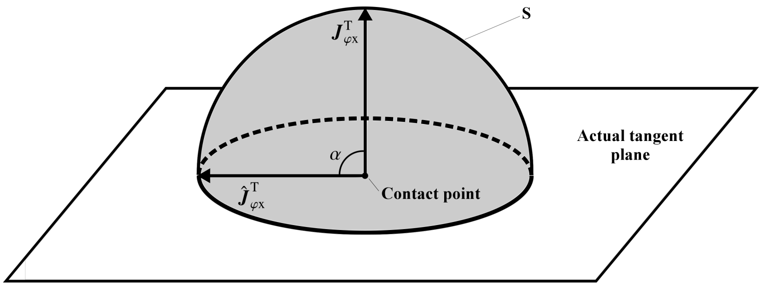

Figure A1. Region $\mathcal{S}$

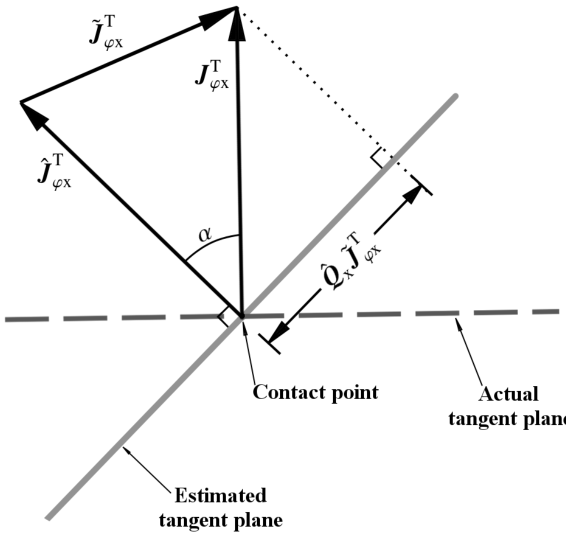

Figure A2. Projections

$0 \leq \alpha \leq \pi/2$ (see Figure A1). Notice that this implies that the region $\mathcal{S}$ must be taken into account in the definition of the region $\mathcal{D}_{\mathrm{s}}$. As can be seen in Figure A2 (recalling that $\|\hat{\boldsymbol{J}}_{\varphi \mathrm{xs}}\| = \|\boldsymbol{J}_{\varphi \mathrm{xs}}\| = 1$), from the cosine rule one has (for $\alpha \leq \pi/2$)

$$\|\tilde{\boldsymbol{J}}_{\varphi \mathrm{xs}}\|^2 = 2\left(1 - \cos(\alpha)\right) \leq 2\left(1 - \cos^2(\alpha)\right) = 2\|\hat{\boldsymbol{Q}}_{\mathrm{xs}}\tilde{\boldsymbol{J}}^{\mathrm{T}}_{\varphi \mathrm{xs}}\|^2 , \tag{A37}$$

since from the same figure it can be seen that $\|\hat{\boldsymbol{Q}}_{\mathrm{xs}}\tilde{\boldsymbol{J}}^{\mathrm{T}}_{\varphi \mathrm{xs}}\| = \cos(\pi/2 - \alpha) = \sin(\alpha)$. Hence (A37) implies

$$\|\tilde{\boldsymbol{J}}_{\varphi \mathrm{xs}}\| \leq \sqrt{2}\|\hat{\boldsymbol{Q}}_{\mathrm{xs}}\tilde{\boldsymbol{J}}^{\mathrm{T}}_{\varphi \mathrm{xs}}\| . \tag{A38}$$

Moreover, given that the robot always exerts force over the surface, there must exist a constant, say c_λ, such that $\lambda_{\mathrm{s}} \geq c_\lambda > 0, \forall t \geq t_0$. Also, since it has been proven that $\hat{\lambda}_{\mathrm{s}}$ is bounded in $\mathcal{D}_{\mathrm{s}}$, after (18) there must exist a constant, say $c_{\hat{\lambda}}$, such that $0 \leq \hat{\lambda}_{\mathrm{s}} \leq c_{\hat{\lambda}} < \infty, \forall t \geq t_0$. Therefore, (A36) satisfies

$$\dot{V}_{\varphi \mathrm{xs}} \leq -\frac{\gamma c_\lambda}{\hat{\lambda}_{\mathrm{s}} + \epsilon}\|\hat{\boldsymbol{Q}}_{\mathrm{xs}}\tilde{\boldsymbol{J}}^{\mathrm{T}}_{\varphi \mathrm{xs}}\| \left(\|\hat{\boldsymbol{Q}}_{\mathrm{xs}}\tilde{\boldsymbol{J}}^{\mathrm{T}}_{\varphi \mathrm{xs}}\| - \frac{\lambda_{\mathrm{H}}}{c_\lambda}\|\boldsymbol{J}_{\mathrm{s}}^{-\mathrm{T}}(\boldsymbol{q}_{\mathrm{s}})\|\|\tilde{\boldsymbol{z}}_1\| - \frac{c_{\hat{\lambda}} + \epsilon}{\gamma c_\lambda}\sqrt{2}v_{\mathrm{x}} \right) , \tag{A39}$$

where v_{x} is a bound for $\dot{\boldsymbol{J}}_{\varphi \mathrm{xs}}$. By choosing the eigenvalues of $\boldsymbol{A}$ in (49) far away on the left in the complex plane, one guarantees that $(\tilde{\boldsymbol{q}}_{\mathrm{s}}, \ldots, \tilde{\boldsymbol{q}}_{\mathrm{s}}^{(p+2)})$ can be made arbitrarily small independently of the values of k_{vs} and k_{Fis} (see (A29)). Notice that this implies after (45) that $\boldsymbol{f}_{\mathrm{s}}(t)$, and hence $\tilde{\boldsymbol{z}}_1$ in (44), can also be made arbitrarily small, which in turn implies that $(\lambda_{\mathrm{H}}/c_\lambda)\|\boldsymbol{J}_{\mathrm{s}}^{-\mathrm{T}}(\boldsymbol{q}_{\mathrm{s}})\|\|\tilde{\boldsymbol{z}}_1\|$ can be made arbitrarily small as well since after Assumption 1 $\boldsymbol{J}_{\mathrm{s}}^{-1}(\boldsymbol{q}_{\mathrm{s}})$ always exists. Finally, the term $(c_{\hat{\lambda}} + \epsilon/\gamma c_\lambda)\sqrt{2v_{\mathrm{x}}}$ can be made arbitrarily small by setting γ sufficiently large. By defining

$$c_{\mathrm{Q}} \triangleq \frac{\lambda_{\mathrm{H}}}{c_\lambda}\|\boldsymbol{J}_{\mathrm{s}}^{-\mathrm{T}}(\boldsymbol{q}_{\mathrm{s}})\|\|\tilde{\boldsymbol{z}}_1\| + \frac{c_{\hat{\lambda}} + \epsilon}{\gamma c_\lambda}\sqrt{2v_{\mathrm{x}}}\,, \tag{A40}$$

one has

$$\|\hat{\boldsymbol{Q}}_{\mathrm{xs}}\tilde{\boldsymbol{J}}_{\varphi\mathrm{xs}}^{\mathrm{T}}\| \geq c_{\mathrm{Q}} \implies \dot{V}_{\varphi\mathrm{xs}} \leq 0\,. \tag{A41}$$

After (A33), it can be stated

$$\frac{1}{2}\|\tilde{\boldsymbol{J}}_{\varphi\mathrm{xs}}\|^2 \leq V_{\varphi\mathrm{xs}} \leq \frac{1}{2}\|\tilde{\boldsymbol{J}}_{\varphi\mathrm{xs}}\|^2\,, \tag{A42}$$

which means that for $\dot{V}_{\varphi\mathrm{xs}} \leq 0$

$$\|\tilde{\boldsymbol{J}}_{\varphi\mathrm{xs}}(t)\| \leq \|\tilde{\boldsymbol{J}}_{\varphi\mathrm{xs}}(t_0)\|, \quad \forall t \geq t_0\,. \tag{A43}$$

In the extreme case that $\|\hat{\boldsymbol{Q}}_{\mathrm{xs}}\tilde{\boldsymbol{J}}_{\varphi\mathrm{xs}}\| \equiv c_{\mathrm{Q}}$, after (A38), the ultimate bound for $\|\tilde{\boldsymbol{J}}_{\varphi\mathrm{xs}}\|$ is given by

$$\|\tilde{\boldsymbol{J}}_{\varphi\mathrm{xs}}\| \leq \sqrt{2}c_{\mathrm{Q}}\,. \tag{A44}$$

Furthermore, the smaller $\|\hat{\boldsymbol{Q}}_{\mathrm{xs}}\tilde{\boldsymbol{J}}_{\varphi\mathrm{xs}}^{\mathrm{T}}\|$, the smaller $\|\tilde{\boldsymbol{J}}_{\varphi\mathrm{xs}}\|$. By adding the functions defined in (A25), (A31) and (A33) it is obtained the positive definite function

$$\begin{aligned} V = V_{\mathrm{a}} + V_{\mathrm{s}} + V_{\varphi\mathrm{xs}} &= \boldsymbol{x}_{\mathrm{o}}^{\mathrm{T}}\boldsymbol{P}_{\mathrm{o}}\boldsymbol{x}_{\mathrm{o}} + \frac{1}{2}\boldsymbol{s}^{\mathrm{T}}\boldsymbol{H}_{\mathrm{s}}(\boldsymbol{q})\boldsymbol{s} + \frac{1}{4}\frac{k_{\mathrm{Fis}}}{k_{\mathrm{vs}}}(\Delta\bar{F}_{\mathrm{s}})^2 + \frac{1}{2}\tilde{\boldsymbol{J}}_{\varphi\mathrm{xs}}\tilde{\boldsymbol{J}}_{\varphi\mathrm{xs}}^{\mathrm{T}} \\ &= \boldsymbol{y}_{\mathrm{s}}^{\mathrm{T}}\begin{bmatrix} \boldsymbol{P}_{\mathrm{o}} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \frac{1}{2}\boldsymbol{H}_{\mathrm{s}}(\boldsymbol{q}_{\mathrm{s}}) & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \frac{1}{4}\frac{k_{\mathrm{Fis}}}{k_{\mathrm{vs}}} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \frac{1}{2}\boldsymbol{I} \end{bmatrix}\boldsymbol{y}_{\mathrm{s}} = \boldsymbol{y}_{\mathrm{s}}^{\mathrm{T}}\boldsymbol{M}_{\mathrm{s}}(\boldsymbol{q}_{\mathrm{s}})\boldsymbol{y}_{\mathrm{s}}\,, \end{aligned} \tag{A45}$$

where each $\boldsymbol{O}$ is a matrix or vector of zeros of appropriate dimensions. Given Property 2, we can find two positive constants, λ_{m} and λ_{M}, such that

$$\lambda_{\mathrm{m}}\|\boldsymbol{y}_{\mathrm{s}}\|^2 \leq V(\boldsymbol{y}_{\mathrm{s}}) \leq \lambda_{\mathrm{M}}\|\boldsymbol{y}_{\mathrm{s}}\|^2\,. \tag{A46}$$

After (A27), (A32) and (A39), the time derivative of (A45) along the trajectories of the system fulfils

$$\dot{V} \leq -\lambda_{\min}(\boldsymbol{Q}_{\rm o})\|\boldsymbol{x}_{\rm o}\|\left(\|\boldsymbol{x}_{\rm o}\| - \frac{2\lambda_{\max}(\boldsymbol{P}_{\rm o})\|\boldsymbol{B}\|r_{\max}}{\lambda_{\min}(\boldsymbol{Q}_{\rm o})}\right) - k_{\rm vs}\|\boldsymbol{s}\|\left(\|\boldsymbol{s}\| - \frac{c_{\rm a}}{k_{\rm vs}}\right)$$
$$-\frac{1}{4}\frac{k_{\rm Fis}^2 c_\varphi^-}{k_{\rm vs}}|\Delta\bar{F}_{\rm s}|\left(|\Delta\bar{F}_{\rm s}| - \frac{2\lambda_{\rm H}c_\varphi^+}{k_{\rm Fis}c_\varphi^-}\|\tilde{\boldsymbol{z}}_1\|\right)$$
$$-\frac{\gamma c_\lambda}{\hat{\lambda}_{\rm s}+\epsilon}\|\hat{\boldsymbol{Q}}_{\rm xs}\tilde{\boldsymbol{J}}_{\varphi{\rm xs}}^{\rm T}\|\left(\|\hat{\boldsymbol{Q}}_{\rm xs}\tilde{\boldsymbol{J}}_{\varphi{\rm xs}}^{\rm T}\| - \frac{\lambda_{\rm H}}{c_\lambda}\|\boldsymbol{J}_{\rm s}^{-{\rm T}}(\boldsymbol{q}_{\rm s})\|\|\tilde{\boldsymbol{z}}_1\| - \frac{c_{\hat{\lambda}}+\epsilon}{\gamma c_\lambda}\sqrt{2}v_{\rm x}\right). \tag{A47}$$

According to the previous discussion, the terms

$$\frac{2\lambda_{\max}(\boldsymbol{P}_{\rm o})\|\boldsymbol{B}\|r_{\max}}{\lambda_{\min}(\boldsymbol{Q}_{\rm o})},\ \frac{c_{\rm a}}{k_{\rm vs}},\ \frac{2\lambda_{\rm H}c_\varphi^+}{k_{\rm Fis}c_\varphi^-}\|\tilde{\boldsymbol{z}}_1\|,\ \frac{\lambda_{\rm H}}{c_\lambda}\|\boldsymbol{J}_{\rm s}^{-{\rm T}}(\boldsymbol{q}_{\rm s})\|\|\tilde{\boldsymbol{z}}_1\|,\ \text{and}\ \frac{c_{\hat{\lambda}}+\epsilon}{\gamma c_\lambda}\sqrt{2}v_{\rm x}$$

can be made arbitrarily small in $\mathcal{D}_{\rm s}$ by choosing the eigenvalues of $\boldsymbol{A}$ in (49) far away on the left in the complex plane and the gains $k_{\rm vs}$, $k_{\rm Fis}$, and γ large enough, as well as by choosing $y_{\max}$ small enough. Overall, we can always find a positive arbitrarily small constant μ such that

$$\dot{V} \leq 0 \quad \text{if} \quad \|\boldsymbol{y}_{\rm s}\| \geq \mu\,. \tag{A48}$$

Once $\|\boldsymbol{y}_{\rm s}\| = \mu$, from (A46) the maximum value that $\|\boldsymbol{y}_{\rm s}\|$ can take is given by

$$\lambda_{\rm m}\|\boldsymbol{y}_{\rm s}\|^2 \leq V(\boldsymbol{y}_{\rm s}) \leq \lambda_{\rm M}\mu^2 \implies \|\boldsymbol{y}_{\rm s}\| \leq \sqrt{\frac{\lambda_{\rm M}}{\lambda_{\rm m}}}\mu \triangleq b\,, \tag{A49}$$

where b is the ultimate bound of the state $\boldsymbol{y}_{\rm s}$. Recall that it must be guaranteed that $\|\boldsymbol{y}_{\rm s}\| \leq y_{\max}$, $\forall t \geq t_0$. This can be done by setting gains large enough to satisfy

$$\mu < \sqrt{\frac{\lambda_{\rm m}}{\lambda_{\rm M}}}y_{\max}\,. \tag{A50}$$

Also, the initial condition must satisfy

$$\|\boldsymbol{y}_{\rm s}(t_0)\| < \sqrt{\frac{\lambda_{\rm m}}{\lambda_{\rm M}}}y_{\max} \tag{A51}$$

to guarantee that $\boldsymbol{y}_{\rm s}$ never leaves the region $\mathcal{D}_{\rm s}$.

d) Finally, since b can be made arbitrarily small, then $\|\boldsymbol{y}_{\rm s}\|$ can be made arbitrarily close to zero. This implies the that observation errors $\tilde{\boldsymbol{q}}_{\rm s}, \ldots, \tilde{\boldsymbol{q}}_{\rm s}^{(p+2)}$ are made approximately zero. Therefore, after (39), $\tilde{\boldsymbol{q}}_{\rm s2} \approx \boldsymbol{0} \implies \boldsymbol{q}_{\rm s2} \approx \hat{\boldsymbol{q}}_{\rm s2}$, *i.e.,* $\hat{\boldsymbol{q}}_{\rm s2}$ is an arbitrarily close estimation of the vector of joint velocities $\boldsymbol{q}_{\rm s2}$. Also, we have proved that $\tilde{\boldsymbol{z}}_1 \approx \boldsymbol{0}$, which after (9) and (18) means that $\lambda_{\rm s} - \hat{\lambda}_{\rm s} \approx 0 \implies \hat{\lambda}_{\rm s} \approx \lambda_{\rm s}$ for $\lambda_{\rm s} > 0$, which implies arbitrary close estimation of the contact force. Since the ultimate bound of $\|\boldsymbol{y}_{\rm s}\|$, and therefore of $\|\boldsymbol{s}\|$ and $\|\Delta\bar{F}_{\rm s}\|$, can be made arbitrarily small, from (57) one can see that the ultimate bound of $\Delta\lambda_{\rm s}$ must be arbitrarily small as well, implying that force tracking is achieved. △