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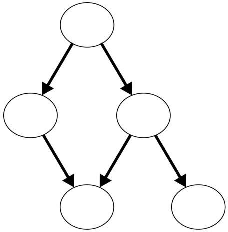
# Bayesian Networks - Inference (Part II)

## Probabilistic Graphical Models

L. Enrique Sucar, INAOE



# Outline

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# Inference - Multiconnected networks

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- There are several classes of algorithms for probabilistic inference on multi conneced BNs:
  - variable elimination,
  - conditioning,
  - junction tree.

# Variable Elimination

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- The variable elimination technique is based on the idea of calculating the probability by marginalizing the joint distribution
- It takes advantage of the independence conditions of the BN and the associative and distributive properties of addition and multiplication to do the calculations more efficiently:
  1. Represent the joint distribution as a product of local probabilities according to the network structure
  2. Summations can be carried out only on the subset of terms which are a function of the variables being normalized

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# Variable Elimination

- Joint probability distribution of $\mathbf{X} = \{X_1, X_2, ..., X_n\}$
- Posterior probability of a certain variable or subset of variables, $X_H$, given a subset of evidence variables, $X_E$; the remaining variables are $X_R$, such that $\mathbf{X} = \{X_H \cup X_E \cup X_R\}$:

$$P(X_H \mid X_E) = P(X_H, X_E)/P(X_E) \quad (1)$$

- We can obtain both terms via marginalization of the joint distribution:

$$P(X_H, X_E) = \sum_{X_R} P(\mathbf{X}) \quad (2)$$

and

$$P(X_E) = \sum_{X_H} P(X_H, X_E) \quad (3)$$

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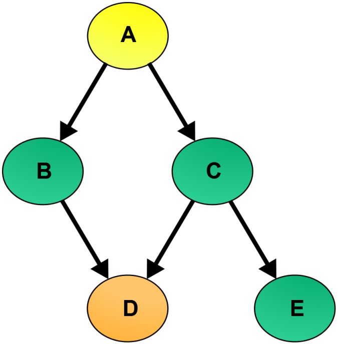
# VE - illustration




- Obtain $P(A \mid D)$ - we need to obtain $P(A, D)$ and $P(D)$.

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# VE - calculations

- *Eliminate* $B, C, E$ from the joint distribution, that is:

$$P(A, D) =$$

$$\sum_B \sum_C \sum_E P(A)P(B \mid A)P(C \mid A)P(D \mid B, C)P(E \mid C)$$

- By *distributing* the summations we can arrive to the following equivalent expression:

$$P(A, D) =$$

$$P(A) \sum_B P(B \mid A) \sum_C P(C \mid A)P(D \mid B, C) \sum_E P(E \mid C)$$

- If all variables are binary, this implies a reduction from 32 operations to 9 operations

# VE - example

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- Obtain $P(E \mid F = f_1) = P(E, F = f_1)/P(F = f_1)$
- Joint distribution:
  $P(C, E, F, D) = P(C)P(E \mid C)P(F \mid E)P(D \mid E)$

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# VE - example

- First calculate $P(E, F)$; by reordering the operations:

$$P(E,F) = \sum_D P(F \mid E) P(D \mid E) \sum_C P(C) P(E \mid C)$$

- Calculation for each value of $E$, given $F = f_1$:

$$P(e_1, f_1) = \sum_D P(f_1 \mid e_1) P(D \mid e_1) \sum_C P(C) P(e_1 \mid C)$$

$$P(e_1, f_1) = \sum_D P(f_1 \mid e_1) P(D \mid e_1) [0.9 \times 0.8 + 0.7 \times 0.2]$$

$$P(e_1, f_1) = \sum_D P(f_1 \mid e_1) P(D \mid e_1) [0.86]$$

$$P(e_1, f_1) = [0.9 \times 0.7 + 0.9 \times 0.3][0.86]$$

$$P(e_1, f_1) = [0.9][0.86] = 0.774$$

# VE - example

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- In a similar way we obtain $P(e_2, f_1)$; and then from these values we can calculate $P(f_1) = \sum_E P(E, f_1)$
- Finally, we calculate the posterior probability of $E$ given $f_1$: $P(e_1 \mid f_1) = P(e_1, f_1)/P(f_1)$ and $P(e_2 \mid f_1) = P(e_2, f_1)/P(f_1)$

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- The critical aspect of the variable elimination algorithm is to select the appropriate order for eliminating each variable, as this has an important effect on the number of required operations
- The different terms that are generated during the calculations are known as *factors* which are functions over a subset of variables, in the previous example, one of the factors is $f(C,E) = P(C)P(E \mid C)$
- The computational complexity in terms of space and time of the variable elimination algorithm is determined by the size of the factors – is exponential on the number of variables in the factor.

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# Elimination Order

- Finding the *best* order is in general a NP-Hard problem
- There are several heuristics that help to determine a *good* ordering for variable elimination:

  Min-degree: eliminate the variable that leads to the smallest possible factor; which is equivalent to eliminating the variable with the smallest number of neighbors in the current elimination graph.

  Min-fill: eliminate the variable that leads to adding the minimum number of edges to the interaction graph.
- These heuristics can be explained based on the *interaction graph* –an undirected graph that is built during the process of variable elimination

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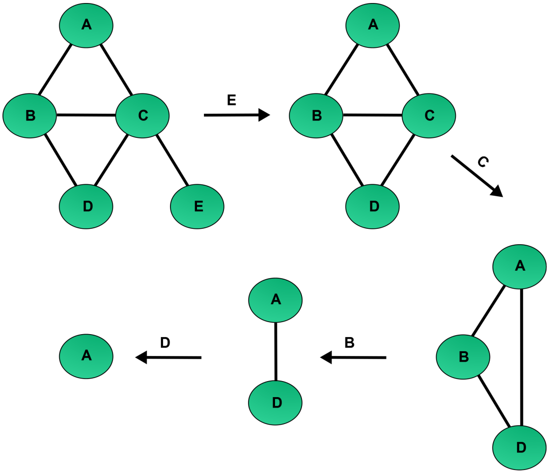
# Interaction graph




- When $X_j$ is eliminated the interaction graph is modified: (i) adding an arc between each pair of neighbors of $X_j$, (ii) deleting variable $X_j$ from the graph

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- A disadvantage of variable elimination is that it only obtains the posterior probability of one variable
- To obtain the posterior probability of each non-instantiated variable in a BN, the calculations have to be repeated for each variable
- Next, we describe two algorithms that calculate the posterior probabilities for all variables at the same time

# Conditioning

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- The conditioning method is based on the fact that an instantiated variable *blocks* the propagation of the evidence in a Bayesian network
- Can *cut* the graph at an instantiated variable, and this can transform a multi connected graph into a polytree, for which we can apply the probability propagation algorithm
- If these variables are not actually known, we can set them to each of their possible values, and then do probability propagation for each value
- With each propagation we obtain a probability for each unknown variable – the final probability values are obtained as a weighted combination of these probabilities

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# Formalization

- Formally, we want to obtain the probability of any variable, $B$, given the evidence $E$, conditioning on variable $A$. By the rule of total probability:

$$P(B \mid E) = \sum_i P(B \mid E, a_i) P(a_i \mid E) \tag{4}$$

- Where:

  $P(B \mid E, a_i)$ is the posterior probability of $B$ which is obtained by probability propagation for each possible value of $A$.

  $P(a_i \mid E)$ is a *weight*.

# Formalization

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- By applying the Bayes rule we obtain the following equation to estimate the weights:

$$P(a_i \mid E) = \alpha P(a_i) P(E \mid a_i) \tag{5}$$

- The first term, $P(a_i)$, can be obtained by propagating without evidence. The second term, $P(E \mid a_i)$, is calculated by propagation with $A = a_i$ to obtain the probability of the evidence variables. $\alpha$ is a normalizing constant

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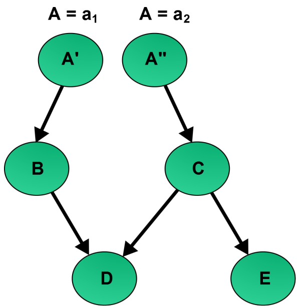
# Example




- If the evidence is $D$, $E$, then probabilities for the other variables, $A$, $B$, $C$ can be obtained via conditioning

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# Example

1. Obtain the prior probability of $A$ (in this case it is already given as it is a root node).
2. Obtain the probability of the evidence nodes $D, E$ for each value of $A$ by propagation in the polytree.
3. Calculate the weights, $P(a_i \mid D, E)$, from (1) and (2) with the Bayes rule.
4. Estimate the probability of $B$ and $C$ for each value of $A$ given the evidence by probability propagation in the polytree.
5. Obtain the posterior probabilities for $B$ and $C$ from (3) and (4) by applying equation 4.

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- In general, to transform a multi connected BN to a polytree we need to instantiate $m$ variables
- Thus, propagation must be performed for all the combinations of values (cross product) of the instantiated variables
- If each variable has $k$ values, then the number of propagations is $k^m$

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# Junction Tree

- The junction tree method is based on a transformation of the BN to a junction tree, where each node in this tree is a group or cluster of variables
- Probabilistic inference is performed over this new representation, via propagation over the junction tree
- The probability of a variable is obtained by marginalization over the "junction" (clique)

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# Transformation

1. Eliminate the directionality of the arcs.
2. Order the nodes in the graph (based on *maximum cardinality*).
3. Moralize the graph (add an arc between pairs of nodes with common children).
4. If necessary add additional arcs to make the graph *triangulated*.
5. Obtain the *cliques* of the graph (subsets of nodes that are fully connected and are not subsets of other fully connected sets).
6. Build a junction tree in which each node is a clique and its parent is any node that contains all common previous variables according to the ordering.

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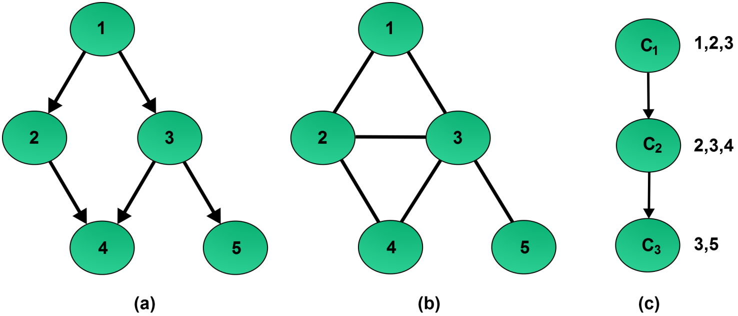
# Transformation - example



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- Once the junction tree is built, inference is based on probability propagation over the junction tree
- Initially the joint probability (potential) of each macro node is obtained, and given some evidence, this is propagated to obtain the posterior probability of each junction
- The individual probability of each variable is obtained from the joint probability of the appropriate junction via marginalization

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# Preprocessing

In the preprocessing phase the potentials of each clique are obtained following the next steps:

1. Determine the set of variables for each clique, $C_i$.
2. Determine the set of variables that are common with the previous (parent) clique, $S_i$.
3. Determine the variables that are in $C_i$ but not in $S_i$: $R_i = C_i - S_i$.
4. Calculate the potential of each clique, $clq_i$, as the product of the corresponding CPTs: $\psi(clq_i) = \prod_j P(X_j \mid Pa(X_j))$; where $X_j$ are the variables in $clq_i$.

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# Preprocessing - example

Cliques: $clq_1 = \{1,2,3\}$, $clq_2 = \{2,3,4\}$, $clq_3 = \{3,5\}$.
Then the preprocessing phase is:

$C$: $C_1 = \{1,2,3\}$, $C_2 = \{2,3,4\}$, $C_3 = \{3,5\}$.

$S$: $S_1 = \emptyset$, $S_2 = \{2,3\}$, $S_3 = \{3\}$.

$R$: $R_1 = \{1,2,3\}$, $R_2 = \{4\}$, $R_3 = \{5\}$.

Potentials: $\psi(clq_1) = P(1)P(2 \mid 1)P(3 \mid 2)$,
$\psi(clq_2) = P(4 \mid 3,2)$, $\psi(clq_3) = P(5 \mid 3)$.

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# Propagation

- The propagation phase proceeds in a similar way to belief propagation for trees, by propagating $\lambda$ messages bottom-up and $\pi$ messages top-down
- **Bottom-Up Propagation**
  1. Calculate the $\lambda$ message to send to the parent clique: $\lambda(C_i) = \sum_R \psi(C_i)$.
  2. Update the potential of each clique with the $\lambda$ messages of its sons: $\psi(C_j)' = \lambda(C_i)\psi(C_j)$.
  3. Repeat the previous two steps until reaching the root clique.
  4. When reaching the root node obtain $P'(C_r) = \psi(C_r)'$.

# Propagation

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- **Top-Down Propagation**
  1. Calculate the $\pi$ message to send to each child node $i$: $\pi(C_i) = \sum_{C_j - S_i} P'(C_j)$.
  2. Update the potential of each clique when receiving the $\pi$ message of its parent: $P'(C_i) = \pi(Ci)\psi(C_i)'$.
  3. Repeat the previous two steps until reaching the leaf nodes in the junction tree.
- When there is evidence, the potentials for each clique are updated based on the evidence, and the same propagation procedure is followed
- Finally, the marginal posterior probabilities of each variable are obtained from the clique potentials via marginalization: $P(X) = \sum_{C_i - X} P'(C_i)$

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# Complexity analysis

- In the worst case, probabilistic inference for Bayesian networks is NP-Hard
- The time and space complexity is determined by what is known as the *tree-width* - a tree-structured BN (maximum one parent per variable) has a tree-width of one. A polytree with at most $k$ parents per node has a tree-width of $k$
- In general, the tree-width is determined by how *dense* the topology of the network is, and this has affects: (i) the size of the largest factor in the variable elimination algorithm; (ii) the number of variables that need to be instantiated in the conditioning algorithm, (iii) the size of the largest clique in the junction tree algorithm

# Loopy Propagation

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- This is simply the application of the probability propagation algorithm for multi connected networks
- Although in this case the conditions for this algorithm are not satisfied, and it only provides an approximate solution
- Given that the BN is not singly connected, as the messages are propagated, these can *loop* through the network

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# Procedure

1. Initialize the $\lambda$ and $\pi$ values for all nodes to random values.
2. Repeat until convergence or a maximum number of iterations:
    1. Do probability propagation according to the algorithm for singly connected networks.
    2. Calculate the posterior probability for each variable.

# Convergence

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- The algorithm converges when the difference between the posterior probabilities for all variables of the current and previous iterations is below a certain threshold
- It has been found empirically that for certain structures this algorithm converges to the true posterior probabilities; however, for other structures it does not converge
- An important application of loopy belief propagation is in “Turbo Codes”; which is a popular error detection and correction scheme used in data communications

# Stochastic simulation

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- Stochastic simulation algorithms consist in *simulating* the BN several times, where each simulation gives a sample value for all non-instantiated variables
- These values are chosen randomly according to the conditional probability of each variable
- This process is repeated $N$ times, and the posterior probability of each variable is approximated in terms of the frequency of each value in the sample space

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# Logic Sampling

- Logic sampling is a basic stochastic simulation algorithm that generates samples according to the following procedure:
  1. Generate sample values for the root nodes of the BN according to their prior probabilities. That is, a random value is generated for each root variable $X$, following a distribution according to $P(X)$.
  2. Generate samples for the next *layer*, that is the sons of the already sampled nodes, according to their conditional probabilities, $P(Y \mid Pa(Y))$, where $Pa(Y)$ are the parents of $Y$.
  3. Repeat (2) until all the leaf nodes are reached.
- The previous procedure is repeated $N$ times to generate $N$ samples. The probability of each variable is estimated as the fraction of times (frequency) that a value occurs in the $N$ samples, that is, $P(X = x_i) \sim No(x_i)/N$

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# Example

10 samples generated by logic sampling (assuming all variables are binary):

| variables | A | B | C | D | E |
|---|---|---|---|---|---|
| $sample_1$ | T | F | F | F | T |
| $sample_2$ | F | T | T | F | F |
| $sample_3$ | T | F | F | T | F |
| $sample_4$ | F | F | T | F | T |
| $sample_5$ | T | F | T | T | F |
| $sample_6$ | F | F | F | F | T |
| $sample_7$ | F | T | T | T | F |
| $sample_8$ | F | F | F | F | F |
| $sample_9$ | F | F | F | T | F |
| $sample_{10}$ | T | T | T | T | F |

# Probabilities - no evidence

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If there is no evidence, then given these samples, the marginal probabilities are estimated as follows:

- $P(A = T) = 4/10 = 0.4$
- $P(B = T) = 3/10 = 0.3$
- $P(C = T) = 5/10 = 0.5$
- $P(D = T) = 5/10 = 0.5$
- $P(E = T) = 3/10 = 0.3$

# Probabilities - with evidence

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In the case where there is evidence with $D = T$, we eliminate all the samples where $D = F$, and estimate the posterior probabilities from the remaining 5 samples:

- $P(A = T \mid D = T) = 3/5 = 0.6$
- $P(B = T \mid D = T) = 2/5 = 0.4$
- $P(C = T \mid D = T) = 3/5 = 0.6$
- $P(E = T \mid D = T) = 1/5 = 0.2$

# Likelihood Weighting

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- A disadvantage of logic sampling when evidence exists is that many samples have to be discarded
- Likelihood weighting generates samples in the same way as logic sampling, however when there is evidence the non-consistent samples are not discarded
- Each sample is given a weight according to the weight of the evidence for this sample

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# Weighting

- Given a sample $s$ and the evidence variables $\mathbf{E} = \{E_1, ..., E_m\}$, the weight of sample $s$ is estimated as:

$$W(\mathbf{E} \mid s) = P(E_1)P(E_2)...P(E_m) \tag{6}$$

where $P(E_i)$ is the probability of the evidence variable $E_i$ for that sample

- The posterior probability for each variable $X$ taking value $x_i$ is estimated by dividing the sum of the weights $W_i(X = x_i)$ for each sample where $X = x_i$ by the total weight for all the samples:

$$P(X = x_i) \sim \sum_i W_i(X = xi) / \sum_i W_i \tag{7}$$

# MPE

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- The *most probable explanation* (MPE) or *abduction* problem consists in determining the most probable values for a subset of variables (explanation subset) in a BN given some evidence
- Two variants: *total abduction* and *partial abduction*
- In the total abduction problem, the explanation subset is the set of all non-instantiated variables
- In partial abduction, the explanation subset is a proper subset of the non-instantiated variables

# Formally

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- Consider the set of variables $\mathbf{X} = \{X_E, X_R, X_H\}$, where $X_E$ is the subset of instantiated variables; then we can formalize the MPE problems as follows:

  Total abduction: $ArgMax_{X_H, X_R} P(X_H, X_R \mid X_E)$.
  Partial abduction: $ArgMax_{X_H} P(X_H \mid X_E)$.

# Solution

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- One way to solve the MPE problem is based on a modified version of the variable elimination algorithm
- Total abduction, we substitute the summations by maximizations:

$$max_{X_H, X_R} P(X_H, X_R \mid X_E)$$

- Partial abduction, we sum over the variables that are not in the explanation subset and maximize over the explanation subset:

$$max_{X_H} \sum_{X_R} P(X_H, X_R \mid X_E)$$

# Continuous variables

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- When dealing with continuos variables, one option is to discretize them; however, this could result in a loss of information or in an unnecessary increase in computational requirements
- Another alternative is to operate directly on the continuous distributions
- Probabilistic inference techniques have been developed for some distribution families, in particular for Gaussian variables

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# Gaussian variables

The basic algorithm makes the following assumptions:

1. The structure of the network is a polytree.
2. All the sources of uncertainty are Gaussians and uncorrelated.
3. There is a linear relationship between each variable and its parents:

$$X = b_1 U_1 + b_2 U_2 + ... + b_n U_n + W_X$$

Where $U_i$ are parents of variable $X$, $b_i$ are constant coefficients and $W_X$ represents Gaussian noise with a zero mean.

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# Inference

- The inference procedure is analogous to belief propagation in discrete BNs, but instead of propagating probabilities, it propagates means and standard deviations
- The posterior probability of a variable can be written as:

$$P(X \mid E) = N(\mu_X, \sigma_X)$$

  Where $\mu_X$ and $\sigma_X$ are the mean and standard deviation of $X$ given the evidence $E$, respectively.
- We calculate the mean and standard deviation for each variable via a propagation algorithm

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# Propagation

- Each variable sends to its parent variable $i$:

$$\mu_i^- = (1/b_i)[\mu_\lambda - \sum_{k \neq i} b_k \mu_k^+] \tag{8}$$

$$\sigma_i^- = (1/b_i^2)[\sigma_\lambda - \sum_{k \neq i} b_k^2 \sigma_k^+] \tag{9}$$

- Each variable sends to its child node $j$:

$$\mu_j^+ = \frac{\sum_{k \neq j} \mu_k^- / \sigma_k + \mu_\pi / \sigma_\pi]}{\sum_{k \neq} 1/\sigma_k^- + \mu_\pi / \sigma_\pi} \tag{10}$$

$$\sigma_j^+ = [\sum_{k \neq j} 1/\sigma_k^- + 1/\sigma_\pi]^{-1} \tag{11}$$

# Propagation

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- Each variable integrates the messages it receives from its sons and parents via the following equations:

$$\mu_\pi = \sum_i b_i \mu_i^+ \tag{12}$$

$$\sigma_\pi = \sum_i b_i^2 \sigma_i^+ \tag{13}$$

$$\mu_\lambda = \sigma_\lambda \sum_j \mu_j^- / \sigma_j^- \tag{14}$$

$$\sigma_\lambda = [\sum_j 1/\sigma_j^-]^{-1} \tag{15}$$

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- Finally, each variable obtains its mean and standard deviation by combining the information from its parent and children nodes:

$$\mu_X = \frac{\sigma_\pi \mu_\lambda + \sigma_\lambda \mu_\pi}{\sigma_\pi + \sigma_\lambda} \tag{16}$$

$$\sigma_X = \frac{\sigma_\pi \sigma_\lambda}{\sigma_\pi + \sigma_\lambda} \tag{17}$$

- Propagation for other distributions is more difficult, as they do not have the same properties of the Gaussian; in particular, the product of Gaussians is also a Gaussian

# Information validation

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- Many systems use information to make decisions; if this information is erroneous it could lead to non-optimal decisions
- In many applications there are different sources of information, i.e. sensors, which are not independent; the information from one source gives us clues about the other sources
- If we can represent these dependencies between the different sources, then we can use it to detect possible errors and avoid erroneous decisions

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# Algorithm

- The algorithm starts by building a model of the dependencies between sources of information (variables) represented as a Bayesian network
- The validation is done in two phases. In the first phase, potential faults are detected by comparing the actual value with the one predicted from the related variables
- In the second phase, the real faults are isolated by constructing an additional Bayesian network based on the Markov blanket property

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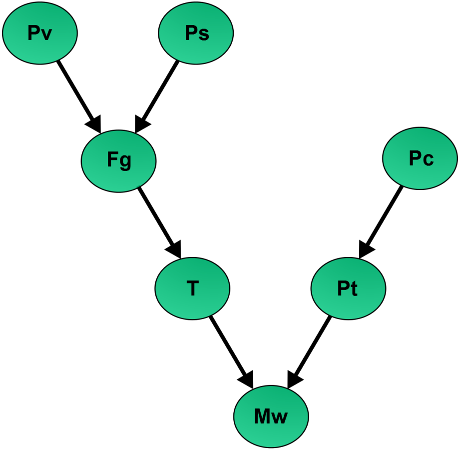
# Fault detection

- Build a probabilistic model relating all the variables in the application domain
- Example - gas turbine:



# Fault detection

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- Suppose it is required to validate the temperature measurements in the turbine
- By reading the values of the rest of the sensors, and applying probability propagation, it is possible to calculate a posterior probability distribution of the temperature given all the evidence, i.e., $P(T \mid Mw, P, Fg, Pc, Pv, Ps)$
- If the real observed value coincides with a *valid* value – that has a *high* probability, then the sensor is considered correct; otherwise it is considered faulty

# Fault detection

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- If a validation of a single sensor is made using a faulty sensor, then a faulty validation can be expected
- In the example above, what happens if $T$ is validated using a faulty $Mw$ sensor?
- By applying this validation procedure, we may only detect a faulty condition, but we are not able to identify which is the real faulty sensor – *apparent fault*
- An isolation stage is needed

# Fault isolation

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- The isolation phase is based on the *Markov Blanket* (MB) property
- The *Extended Markov Blanket* of a node $X$ ($EMB(X)$) as the set of sensors formed by the sensor itself plus its MB
- For example, $EMB(Fg) = \{Fg, Pv, Ps, T\}$
- Utilizing this property, if a fault exists in one of the sensors, it will be revealed in all of the sensors in its EMB. On the contrary, if a fault exists outside a sensors' EMB, it will not affect the estimation of that sensor
- The EMB is used to create a *fault isolation* module that distinguishes the *real faults* from the apparent faults

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# Fault isolation theory

1. If $S = \phi$ there are no faults.
2. If $S$ is equal to the EMB of a sensor $X$, and there is no other EMB which is a subset of $S$, then there is a *single real fault* in $X$.
3. If $S$ is equal to the EMB of a sensor $X$, and there are one or more EMBs which are subsets of $S$, then there is a real fault in $X$, and possibly, real faults in the sensors whose EMBs are subsets of $S$. In this case, there are possibly *multiple indistinguishable* real faults.
4. If $S$ is equal to the union of several EMBs and the combination is unique, then there are *multiple distinguishable* real faults in all the sensors whose EMB are in $S$.
5. If none of the above cases is satisfied, then there are multiple faults but they can not be distinguished

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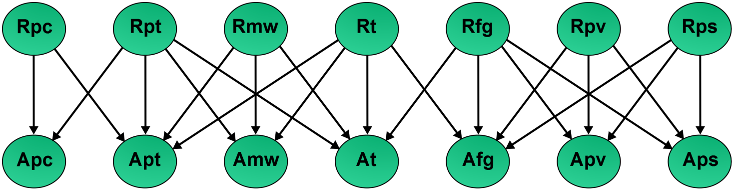
# Isolation network

- The isolation network is formed by two levels:
    - The root nodes represent the real faults, where there is one per sensor or variable
    - The lower level is formed by one node representing the apparent fault for each variable. Notice that the arcs are defined by the EMB of each variable



# Reliability analysis

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- In the reliability analysis of a complex system, a common approach is to divide the system into smaller elements, units, subsystems, or components
- This subdivision generates a "block diagram" that is similar to the description of the system in operation
- For each element, the *failure rate* is specified, and based on these, the reliability of the complete system is obtained
- Traditional techniques assume that faults are independent

# Reliability modeling with BN

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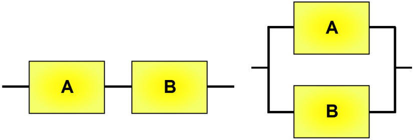
- In a block diagram representation there are two basic structures: serial and parallel components
- A serial structure implies that the two components should operate correctly for the system to function
- In parallel structures, it is sufficient for one of the components to operate for the system to function




(a) (b)

# Basic structures

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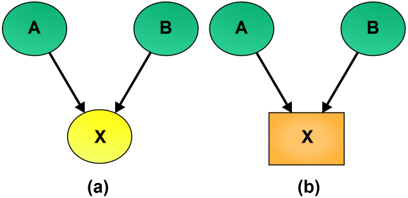
- The basic series and parallel block diagrams can be represented with a Bayesian network
- The structure is the same in both cases, the difference is the conditional probability matrix



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- Series (AND):

| X | $A, B$ | $A, \neg B$ | $\neg A, B$ | $\neg A, \neg B$ |
|---|---|---|---|---|
| Success | 1 | 0 | 0 | 0 |
| Failure | 0 | 1 | 1 | 1 |

- Parallel (OR):

| X | $A, B$ | $A, \neg B$ | $\neg A, B$ | $\neg A, \neg B$ |
|---|---|---|---|---|
| Success | 1 | 1 | 1 | 0 |
| Failure | 0 | 0 | 0 | 1 |

# Extending the basic models

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- The BN representation of the basic serial/parallel cases can be directly generalized to represent any block diagram that can be reduced to a set of serial and parallel combinations of components
- There are some structures that can not be decomposed to a serial/parallel combination, such as a *bridge*. However, it is also possible to model these cases using BNs

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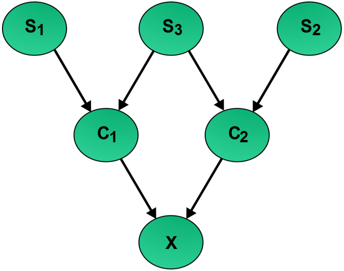
# Example

- Suppose that a system has two components that are affected by three possible failure sources. Source $S_1$ affects component $C_1$, source $S_2$ affects component $C_2$, and source $S_3$ affects both components (common cause)
- In the BN, the CPT for all three non root nodes ($C_1$, $C_2$, $X$) is equivalent to that of a serial component combination



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