Submodular function maximization

Hugo Jair Escalante

Outline

- Set functions
- Submodularity and monotonicity
- Maximization of submodular functions
- Applications: CSMMI
- Dealing with streams
- Experiments & results
- Discussion and final remarks

Set functions

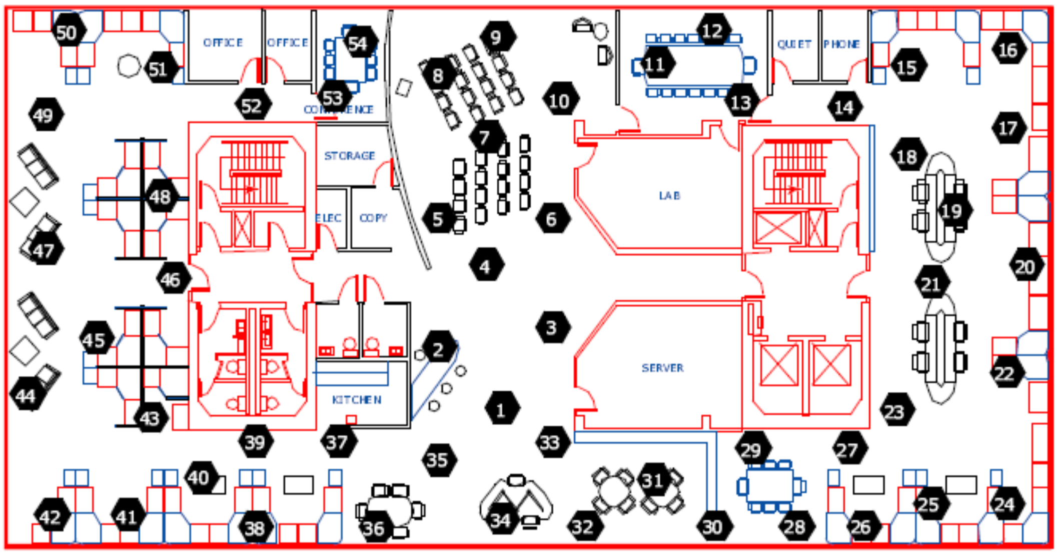
- Functions of the form:

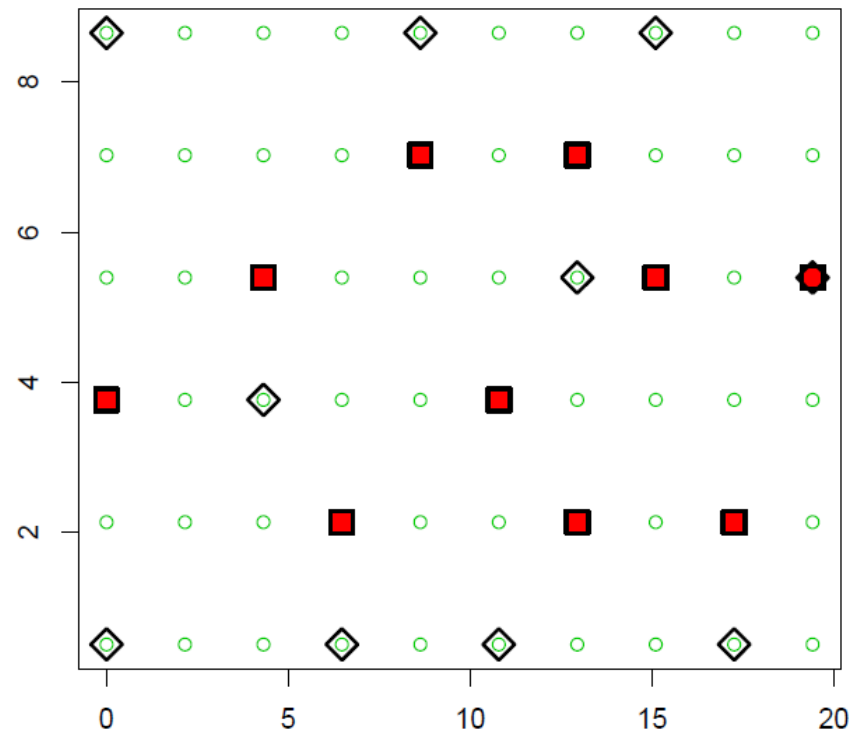
$$f : 2^V \rightarrow \mathbb{R}$$

With V a finite set, and assuming $f(\Phi)=0$

- **Example:** consider the problem of placing sensors to cover some space represented by locations (V), and $f(S)$ the utility obtained when placing sensors at locations S

Set functions





Submodularity

Definition 1.1 (Discrete derivative) For a set function $f : 2^V \to \mathbb{R}$, $S \subseteq V$, and $e \in V$, let $\Delta_f(e \mid S) := f(S \cup \{e\}) - f(S)$ be the *discrete derivative* of f at S with respect to e.

Definition 1.2 (Submodularity) A function $f : 2^V \to \mathbb{R}$ is *submodular* if for every $A \subseteq B \subseteq V$ and $e \in V \setminus B$ it holds that

$$\Delta(e \mid A) \geq \Delta(e \mid B) .$$

Equivalently, a function $f : 2^V \to \mathbb{R}$ is *submodular* if for every $A, B \subseteq V$,

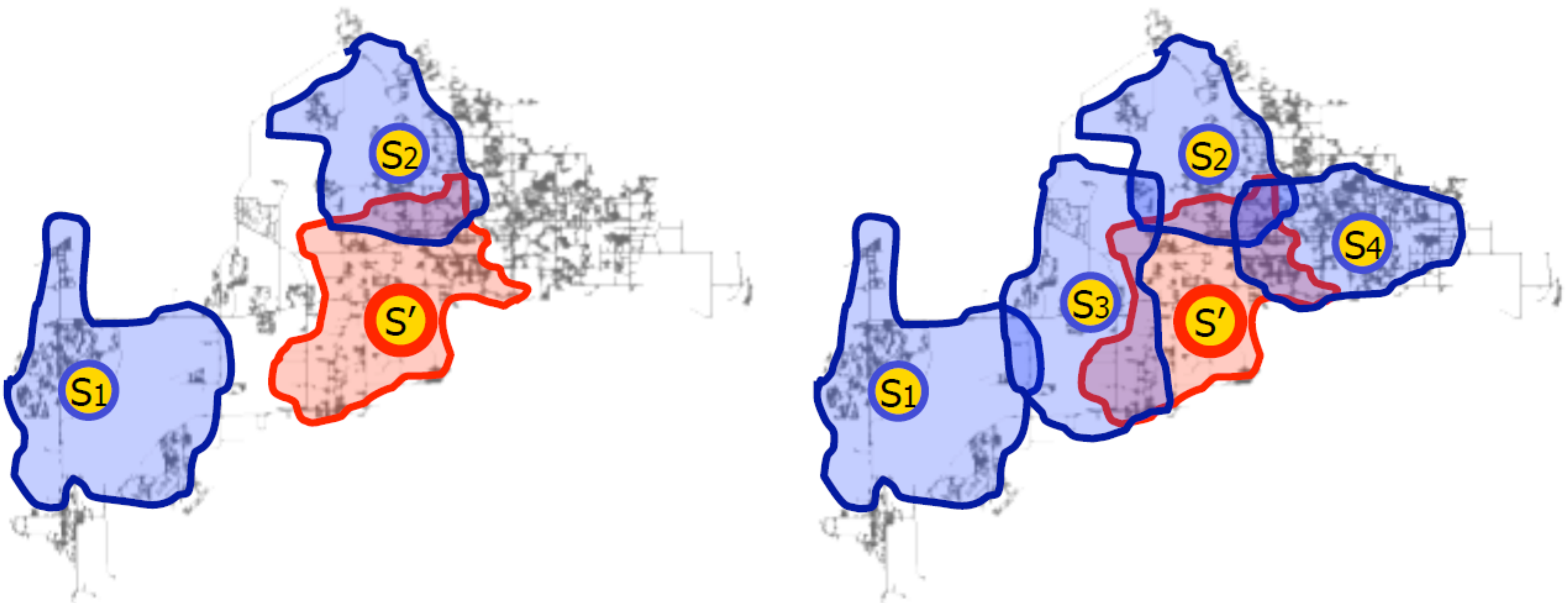
$$f(A \cap B) + f(A \cup B) \leq f(A) + f(B).$$

Submodularity

Definition 1.3 (Monotonicity) A function $f : 2^V \to \mathbb{R}$ is *monotone* if for every $A \subseteq B \subseteq V$, $f(A) \leq f(B)$.

f is called monotone iff for all e and S it holds that: $\triangle_f(e|S) \geq 0$

Submodular functions



(a) *Adding s' to set $\{s_1, s_2\}$*

(b) *Adding s' to superset $\{s_1, \ldots, s_4\}$*

$$\Delta(s' \mid \{s_1, s_2\}) \geq \Delta(s' \mid \{s_1, \ldots, s_4\})$$

Diminishing returns effect in the problem of placing sensors in a water distribution network to detect contaminants: *if more sensors are already placed there is more overlap, and less gain utility. Selecting any given element earlier helps more than selecting it later.*

Submodular functions

- Some submodular functions:
  - Modular functions

    for all $A, B \subseteq V$ it holds that $f(A) + f(B) = f(A \cup B) + f(A \cap B)$.

  - Weighted coverage functions

    $$f(S) := g\Big(\bigcup_{v \in S} v\Big) = \sum_{x \in \bigcup_{v \in S} v} w(x),$$

  - Facility location

    $$f(S) = \sum_{i=1}^{m} \max_{j \in S} M_{i,j}.$$

  - Entropy

    $$H(\mathbf{X}_S) = -\sum_{\mathbf{x}_S} P(\mathbf{x}_S) \log_2 P(\mathbf{x}_S)$$

  - Mutual information

    $$f(S) = I(\mathbf{Y}; \mathbf{X}_S) = H(\mathbf{Y}) - H(\mathbf{Y} \mid \mathbf{X}_S)$$

Maximization of submodular functions

- Submodular functions arise in several domains and problems (e.g., max-coverage, facility location, mutual information, …)

- Consider the problem of **data summarization**: *selecting representative subsets of manageable size out of large data sets*
  - *Exemplar-based clustering, document and corpus summarization, recomender systems, active set selection*

Maximization of submodular functions

- Many data summarization tasks can be formulated as:

$$\max_{S \subseteq V} f(S) \quad \text{s.t.} \quad |S| \leq k$$

- Let S* denote the optimal solution, with value:

$$\texttt{OPT} = f(S^*)$$

- This problem is NP-hard for many classes of submodular functions.

f(*S*) a black-box utility function, for now

Maximization of submodular functions

- Nemhauser et al. showed that a simple greedy (polynomial time) algorithm is highly effective:

  – Start with the empty set $S_0=\Phi$

  – Iterate k-times over the whole data set:

  $$S_i = S_{i-1} \cup \{\arg\max_{e \in V} \Delta_f(e|S_{i-1})\}$$

- Nemhauser et al. proved that: $f(S^g) \geq (1 - 1/e)\mathtt{OPT}$

  (the solution is provably within 63% of the optimal?)

For several classes of monotone submodular functions it is known that this is the best approximation guarantee that one can hope.

Maximization of submodular functions

- Proof: Nemhauser et al. 1978

Many data summarization

- Sample applications:
  - Outbreak detection
  - News article recommendation
  - Non-parametric learning
  - Document and corpus summarization
  - Network inference
  - Viral marketing
  - …

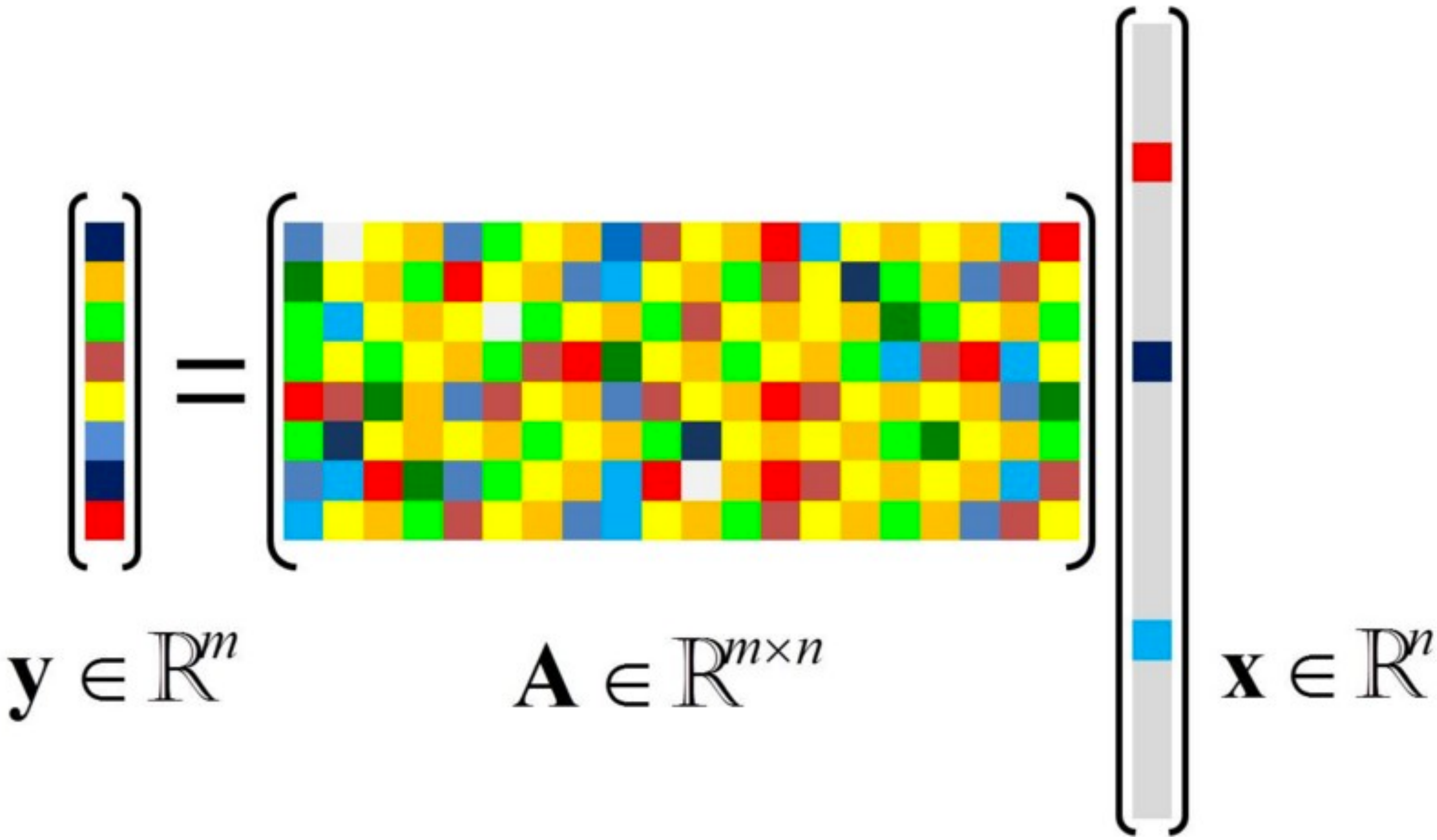
Application: CSMMI

- **Problem:** to learn a dictionary of codewords to be used for the SC representation of videos for action and gesture recognition

Jun Wan, Vassilis Athitsos, Pat Jangyodsuk, Hugo Jair Escalante, Qiuqi Ruan, Isabelle Guyon. **CSMMI: Class-Specific Maximization of Mutual Information for Action and Gesture Recognition.** IEEE Trans. Image Processing, Vol 23(7):3152--3165

Application: CSMMI

- Sparse representations: Idea, to learn a dictionary



Application: CSMMI

- Sparse representations: Idea, to learn a dictionary

$$\Phi_i^0 = [\phi_1, \ldots \phi_j, \ldots \phi_K], \phi_j \in \Re^n$$

- Such that:

$$\min_{\Phi_i^0, X_{\Phi_i^0}} \{\|Y_i - \Phi_i^0 X_{\Phi_i^0}\|_F^2\} \quad s.t. \ \|x_j\|_0 \leq T$$

Application: CSMMI

- The reconstruction error with class specific dictionaries is used as classifier

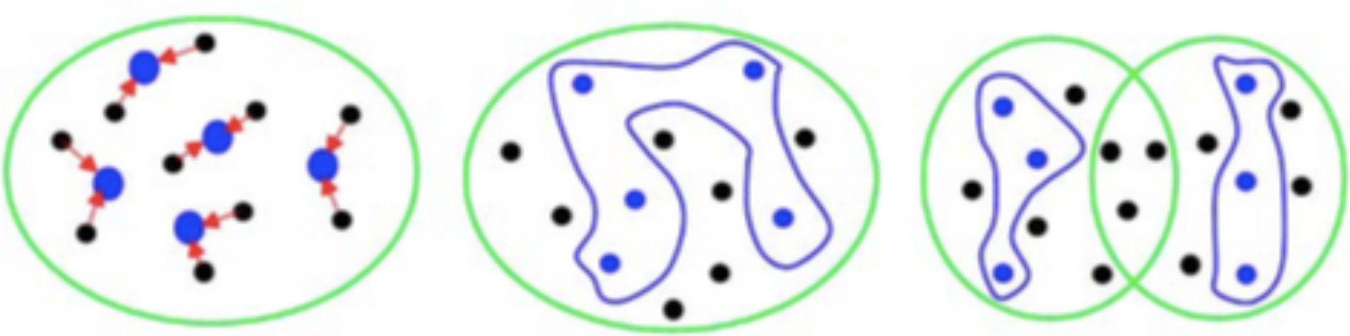
$$i_Y = \underset{i \in [1,2,\ldots,C]}{\arg\min} \|Y - \Phi_i^* \widehat{X_{Y_i}}\|_2^2$$

$$\widehat{X_{Y_i}} = \underset{X_{Y_i}}{\arg\min} \|Y - \Phi_i^* X_{Y_i}\|_F^2 \quad s.t. \ \|x\|_0 \le T$$

Application: CSMMI

- Related works, learn dictionaries regardless of the classes (starting from KSVD)

- CSMMI: to learn class specific dictionaries, maximizing MI



(a) (b) (c)

Application: CSMMI

- Key idea:

1. Learn class specific dictionaries of size K (KSVD)

$$\Phi_i^0 = [\phi_1, \ldots \phi_j, \ldots \phi_K], \phi_j \in \Re^n$$

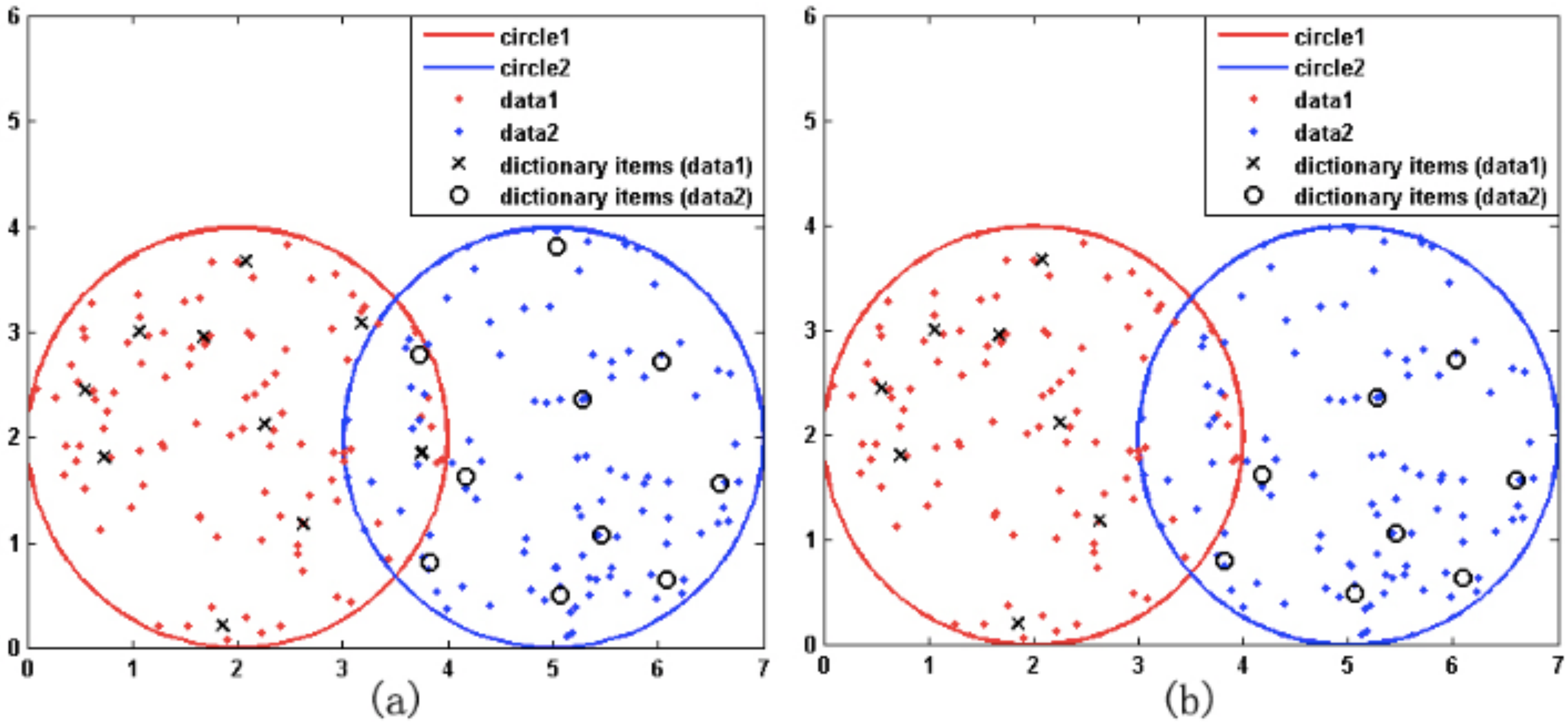
2. Select a subset of codewords , k<K for each class

$$\Phi_1^*, \Phi_2^*, \ldots, \Phi_C^* \; (|\Phi_i^*| = k, \; k < K).$$

$$\arg\max_{\phi_i \in \Phi_i^0 \setminus \Phi_i^*} \overbrace{I(\Phi_i^* \cup \phi_i; \Phi_i^0 \setminus (\Phi_i^* \cup \phi_i)) - I(\Phi_i^*; \Phi_i^0 \setminus \Phi_i^*)}^{intra-class\ MI\ term(\tau_1)}$$

$$-[\overbrace{I(\Phi_i^* \cup \phi_i; \Phi^0 \setminus \Phi_i^0) - I(\Phi_i^*; \Phi^0 \setminus \Phi_i^0)}^{inter-class\ MI\ term(\tau_2)}] \quad (3)$$

Application: CSMMI



Application: CSMMI

- Experimental results

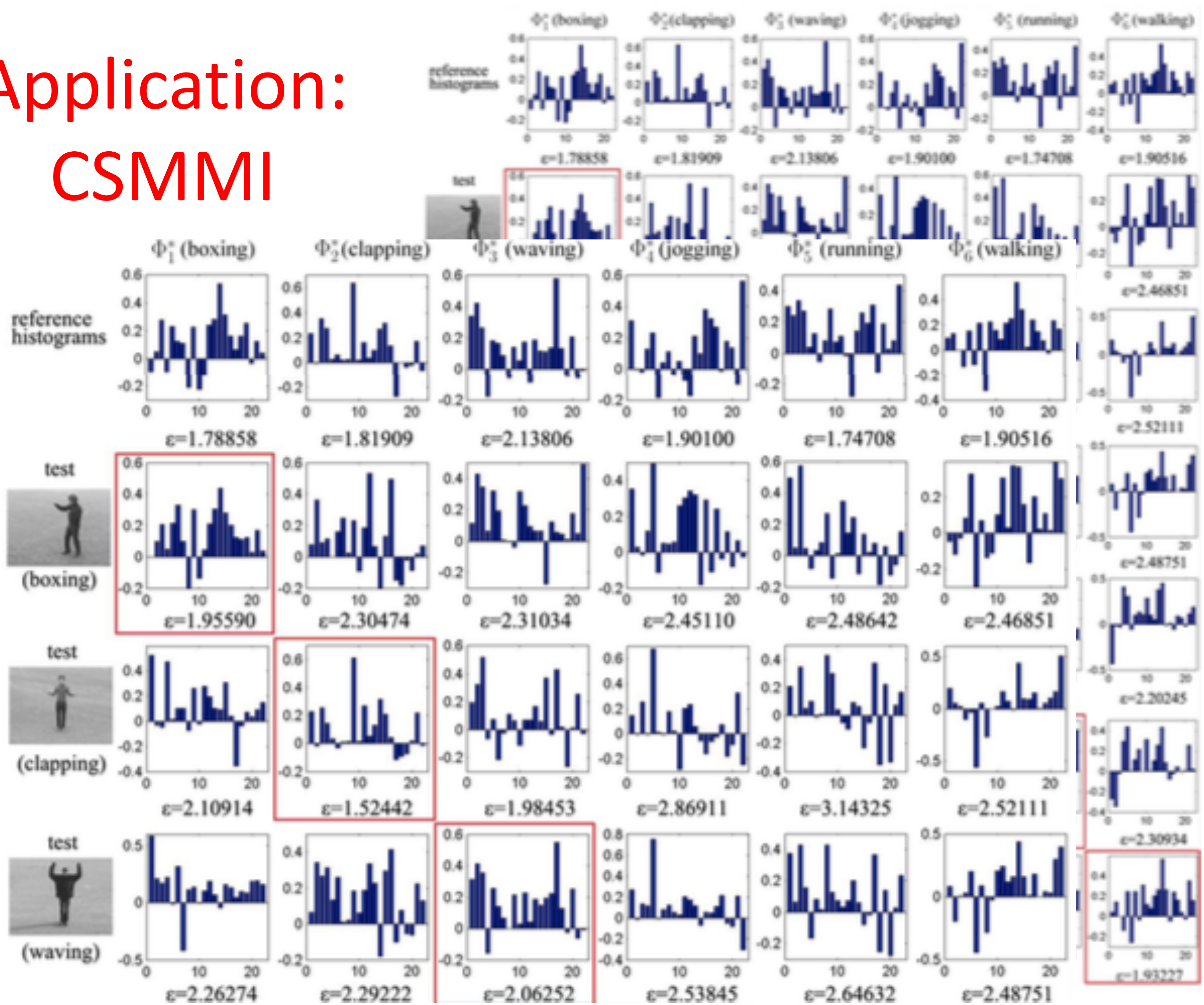
COMPARISON ON THE WEIZMANN ACTION DATASET

Papers	Methods	Dictionary Size	Average Accuracy
[38]	Space-time shape	—	97.83%
[39]	Multiple instance learning +kinematic feature	—	95.75%
[37]	Sparse linear approximation + feature covariance matrices	—	100%
[26]	prototype trees	—	100%
[22]	pLSA+cuboid	1200	90%
[1]	Concatenated dictionary +LMP	256	98.9%
[40]	Self-Similarities	—	95.3%
our method	*CSMMI*+STIP	140	**100%**

COMPARISON ON THE KTH ACTION DATASET

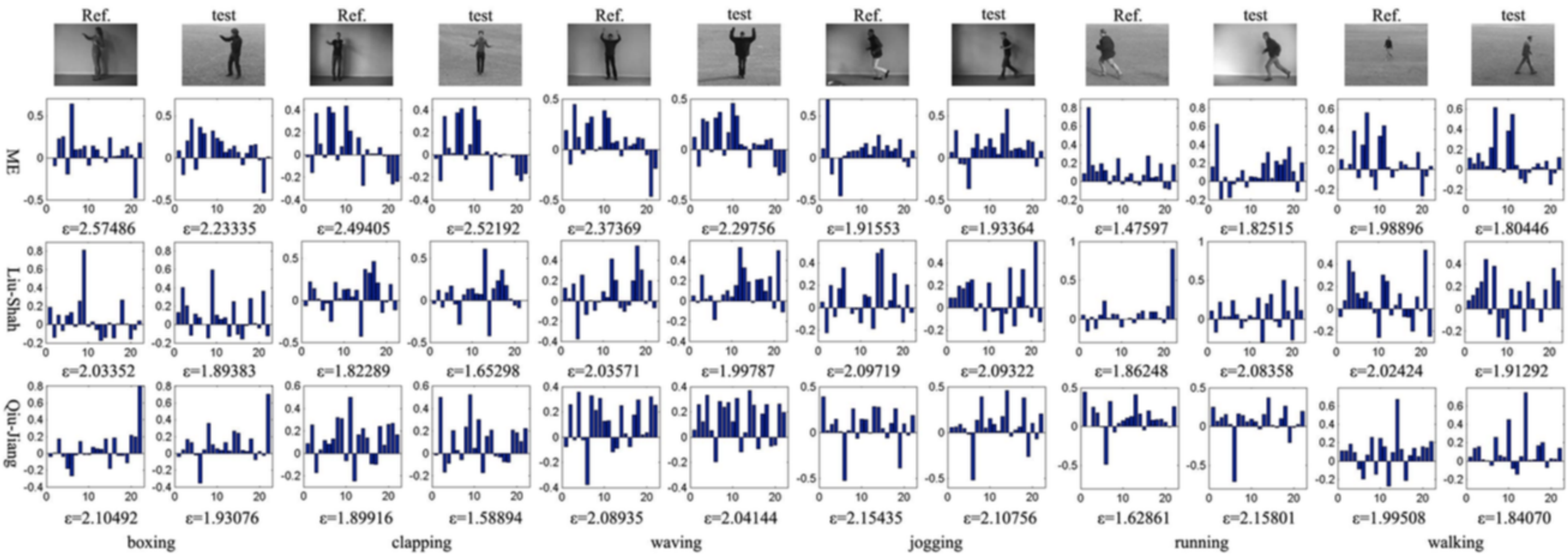
Papers	Methods	Dictionary Size	Average Accuracy
[29]	non-linear SVM+STIP	4000	91.8%
[39]	multiple instance learning +kinematic feature	—	87.7%
[41]	probabilistic spatiotemporal voting	—	88.0%
[37]	sparse linear approximation +Feature Covariance Matrices	—	97.4%
[26]	prototype trees	—	95.77%
[42]	Independent subspace analysis	—	93.9%
our method	*CSMMI*+STIP	365	**98.83%**

Application: CSMMI

- 

Application: CSMMI

- Experimental results



Application: CSMMI

- Experimental results

TABLE III

COMPARISON ON THE UCF SPORTS ACTION DATASET

Papers	Methods	Dictionary Size	Average Accuracy
[30]	Maximum Average Correlation Height	—	69.2%
[42]	Independent subspace analysis	—	86.5%
[1]	class-specific dictionary +cuboid	256	83.8%
[43]	hierarchy of discriminative shape and motion features	300	87.27%
[44]	hough transform-based voting	—	86.6%
[3]	*ME*+STIP	325	81.33%
[6]	*Liu-Shah*+STIP	250	84%
[3]	*Qiu-Jiang*+STIP	308	85.33%
our method	*CSMMI*+STIP	469/250	**98.0%**/87.33%

TABLE IV

COMPARISON ON THE UCF YOUTUBE ACTION DATASET

Papers	Methods	Dictionary Size	Average Accuracy
[31]	cuboid+difussion maps	1000	70.4%
[45]	hybrid features	2000	71.2%
[42]	Independent Subspace Analysis	—	75.8%
[3]	ME+STIP	715	71.1%
[6]	*Liu-Shah*+STIP	624	72.7%
[3]	*Qiu-Jiang*+STIP	678	73.3%
our method	*CSMMI*+STIP	721	**78.6%**

TABLE V

COMPARISON ON THE HOLLYWOOD2 ACTION DATASET

Papers	Methods	Dictionary Size	mAP
[47]	dense+HOG/HOF	4000	47.4%
[48]	dense trajectories	—	58.3%
[42]	independent subspace analysis	—	53.3%
[46]	compensated descriptors +VLAD representation	—	**62.5%**
[3]	ME+STIP	329	41.3%
[6]	*Liu-Shah*+STIP	415	41.9%
[3]	*Qiu-Jiang*+STIP	394	43.2%
our method	*CSMMI*+STIP	437	**62.1%**

TABLE VI

COMPARISON ON THE KECK ACTION DATASET

Papers	Methods	Static setting	Dynamic setting
[26]	prototype trees	95.2%	91.07%
[49]	Product Manifolds	94.4%	92.3%
[3]	*ME*+shape-motion	91.2%	89.3%
[6]	*Liu-Shah*+shape-motion	94.2%	90.7%
[3]	*Qiu-Jiang*+shape-motion	94.9	92.7%
[3]	*Qiu-Jiang**+shape-motion	97%	—
our method	*CSMMI*+shape-motion	**95.1%**	**93.2%**

Many data summarization

- In many contemporary applications, running the standard greedy algorithm is computationally prohibitive:
  - The data set does not fit in main memory
  - Data itself arrives in a stream, possibly cannot be stored

- **Streaming algorithms:** Access only a small fraction of data at any point in time and provide approximate solutions

Streaming submodular maximization algorithms

Streaming submodular maximization

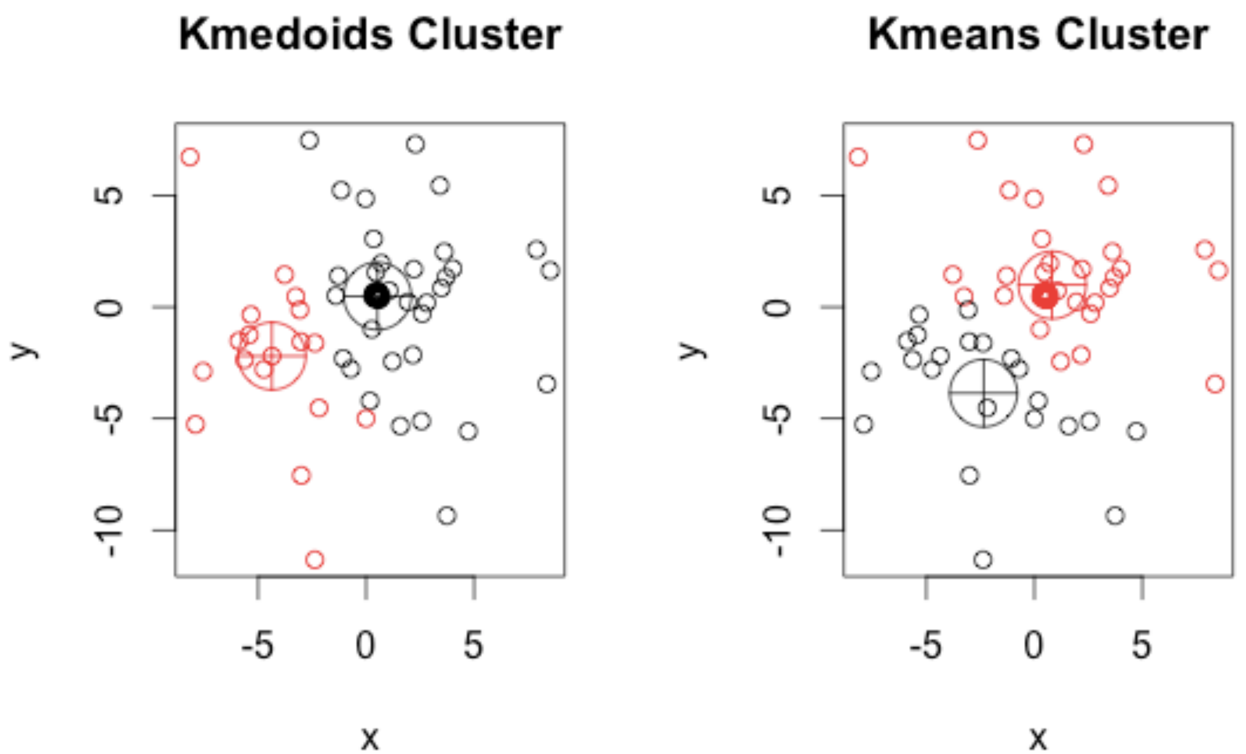
- Assumptions:
  - The ground set is ordered (arbitrarily) and any streaming algorithm must process V in the given order $V = \{e_1, \ldots, e_n\}$
  - At each iteration t the algorithm may maintain a memory $M_t \subset V$ of points and must be ready to output a candidate feasible solution $S_t \subset M_t$ of size at most $|S_t| \leq k$
  - When a new point arrives from the stream, the algorithm may elect to remember it

Streaming submodular maximization

- The performance of a streaming algorithm is measured by:
  - Number of passes the algorithm has to make over the stream
  - Memory required by the algorithm
  - Running time of the algorithm
  - Approximation ratio: $f(S_T)/\mathtt{OPT}$

Sample applications (1)

- **Exemplar based clustering**: Select a set of exemplars that better represent a massive data set.



Sample applications (1)

- **Exemplar based clustering**: Select a set of exemplars that better represent a massive data set.

- K-medoid problem: $L(S) = \frac{1}{|V|} \sum_{e \in V} \min_{v \in S} d(e, v).$

- Introducing an auxiliary element e_0 we can turn L into a monotone submodular function

$$f(S) = L(\{e_0\}) - L(S \cup \{e_0\}).$$

Sample applications (2)

- **Large-scale nonparametric learning** (Active set selection): select a small representative subset of instances and only work with a kernel matrix restricted to this subset

$$K_{V,V} = \begin{pmatrix} \mathcal{K}_{e_1,e_1} & \cdots & \mathcal{K}_{e_1,e_n} \\ \vdots & & \vdots \\ \mathcal{K}_{e_n,e_1} & \cdots & \mathcal{K}_{e_n,e_n} \end{pmatrix}.$$

Sample applications (2)

- The informative vector machine criterion for Gaussian processes

$$f(S) = \frac{1}{2} \log \det(\mathbf{I} + \sigma^{-2} \Sigma_{S,S}),$$

$$f(S) = I(\mathbf{Y}_S; \mathbf{X}_V) = H(\mathbf{X}_V) - H(\mathbf{X}_V | \mathbf{Y}_S) = \frac{1}{2} \log \det(\mathbf{I} + \sigma^{-2} \Sigma_{S,S})$$

$$K_{V,V} = \begin{pmatrix} \mathcal{K}_{e_1,e_1} & \cdots & \mathcal{K}_{e_1,e_n} \\ \vdots & & \vdots \\ \mathcal{K}_{e_n,e_1} & \cdots & \mathcal{K}_{e_n,e_n} \end{pmatrix}.$$

Many data summarization

- Naïve approximations:
  - Greedy algorithm: too comp. expensive and cannot run in real streams
  - Maintaining in memory the k-best elements: performance degrades arbitrarily with k

	# passes	approx. guarantee	memory	update time
Standard Greedy [27]	$O(k)$	$(1-1/e)$	$O(k)$	$O(k)$
Greedy-Scaling [20]	$O(1/\delta)$	$\delta/2$	$kn^{\delta}\log n$	?
Stream-Greedy [14]	multiple	$(1/2-\varepsilon)$	$O(k)$	$O(k)$
Sieve-Streaming	1	$(1/2-\varepsilon)$	$O(k\log(k)/\varepsilon)$	$O(\log(k)/\varepsilon)$

The Sieve-streaming algorithm

- Key observations:
  - Knowing $\mathtt{OPT}$ helps
  - Knowing $m = \max_{e \in V} f(\{e\})$ is enough
  - Lazy updates (approximate m)

The Sieve-streaming algorithm

$$\boxed{S_i = S_{i-1} \cup \{\arg\max_{e \in V} \triangle_f(e|S_{i-1})\}}$$

- Knowing $\texttt{OPT}$ helps.
  - If S_i is the set of the first i elements picked by the greedy algorithm, then the marginal value:

$$\triangle_f(e_{i+1}|S_i)$$

  - Of the next element added is at least:

$$(\texttt{OPT} - f(S_i))/k$$

- **Idea:** identify elements with similarly high marginal value, under a lowered threshold:

$$\beta\texttt{OPT}/k$$

The Sieve-streaming algorithm

$$\boxed{S_i = S_{i-1} \cup \{\arg\max_{e \in V} \triangle_f(e|S_{i-1})\}}$$

- Suppose we know $\texttt{OPT}$ up to a constant factor α, i.e., we have a value *v* such that:

$$\texttt{OPT} \geq v \geq \alpha \cdot \texttt{OPT} \qquad 0 \leq \alpha \leq 1$$

- The algorithm starts with $S_0=\Phi$, and then after observing each element, it adds it to S if the marginal value is at least:

$$(v/2 - f(S))/(k - |S|)$$

and we are still below the cardinality constraint

The Sieve-streaming algorithm

Algorithm 1 SIEVE-STREAMING-KNOW-OPT-VAL

Input: v such that $\texttt{OPT} \geq v \geq \alpha\ \texttt{OPT}$

1: $S = \emptyset$
2: **for** $i = 1$ to n **do**
3:     **if** $\triangle_f(e_i \mid S) \geq \frac{v/2 - f(S)}{k - |S|}$ and $|S_v| < k$ **then**
4:         $S := S \cup \{e_i\}$
5: **return** S

PROPOSITION 5.1. *Assuming input v to algorithm 1 satisfies $\mathit{OPT} \geq v \geq \alpha\ \mathit{OPT}$, the algorithm satisfies the following properties*

- *It outputs a set S such that $|S| \leq k$ and $f(S) \geq \frac{\alpha}{2}\mathit{OPT}$*
- *It does 1 pass over the data set, stores at most k elements and has $O(1)$ update time per element.*

The Sieve-streaming algorithm

- Obtaining a good approximation to $\texttt{OPT}$ is not straightforward

Ain't a very useful estimate!

with *v=km* and *α=1/k*, we get a guarantee:

$$\texttt{OPT}/2k$$

$$m \leq \texttt{OPT} \leq k \cdot m.$$

Equivalently, a function $f : 2^V \to \mathbb{R}$ is *submodular* if for every $A, B \subseteq V$,

$$f(A \cap B) + f(A \cup B) \leq f(A) + f(B).$$

The Sieve-streaming algorithm

- Idea: refining the threshold. Consider the set:

$$O = \{(1+\epsilon)^i | i \in \mathbb{Z}, m \leq (1+\epsilon)^i \leq k \cdot m\}$$

- There should exist at least some $v \in O$ such that: $(1-\epsilon)\texttt{OPT} \leq v \leq \texttt{OPT}$

The Sieve-streaming algorithm

Algorithm 2 SIEVE-STREAMING-KNOW-MAX-VAL

Input: $m = \max_{e \in V} f(\{e\})$

$O = \{(1+\epsilon)^i | i \in \mathbb{Z}, m \leq (1+\epsilon)^i \leq k \cdot m\}$

For each $v \in O$, $S_v := \emptyset$

for $i = 1$ to n **do**

    for $v \in O$ **do**

        if $\triangle_f(e_i \mid S_v) \geq \frac{v/2 - f(S_v)}{k - |S_v|}$ and $|S_v| < k$ **then**

            $S_v := S_v \cup \{e_i\}$

return $\operatorname{argmax}_{v \in O_n} f(S_v)$

PROPOSITION 5.2. *Assuming input m to Algorithm 2 satisfies $m = \max_{e \in V} f(\{e\})$, the algorithm satisfies the following properties*

- *It outputs a set S such that $|S| \leq k$ and $f(S) \geq \left(\frac{1}{2} - \epsilon\right)$ OPT*
- *It does 1 pass over the data set, stores at most $O\left(\frac{k \log k}{\epsilon}\right)$ elements and has $O\left(\frac{\log k}{\epsilon}\right)$ update time per element.*

The Sieve-streaming algorithm

- Final algorithm: relax the assumption we need to know the maximum value of all singletons:
  - Maintain an auxiliary variable *m* which holds the current maximum singleton element
  - Initiate thresholds for an increased range:

$$v = (1+\epsilon)^i, m \leq (1+\epsilon)^i \leq 2 \cdot k \cdot m$$

The Sieve-streaming algorithm

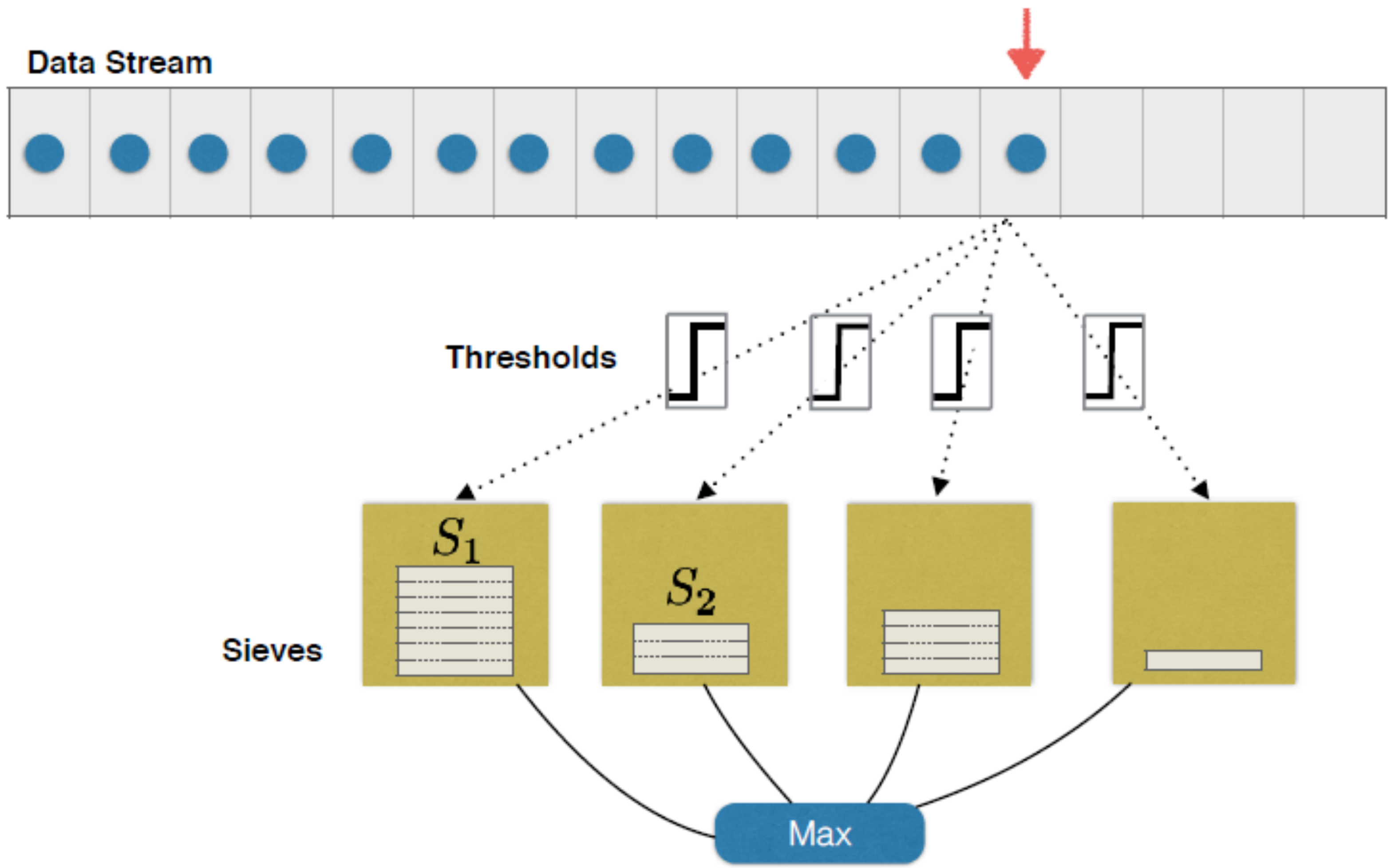
Algorithm 3 SIEVE-STREAMING

$O = \{(1+\epsilon)^i | i \in \mathbb{Z}\}$
For each $v \in O$, $S_v := \emptyset$ (maintain the sets only for the necessary v's lazily)
$m := 0$
for $i = 1$ to n **do**
    $m := \max(m, f(\{e_i\}))$
    $O_i = \{(1+\epsilon)^i | m \leq (1+\epsilon)^i \leq 2 \cdot k \cdot m\}$
    Delete all S_v such that $v \notin O_i$.
    for $v \in O_i$ **do**
        if $\triangle_f(e_i \mid S_v) \geq \frac{v/2 - f(S_v)}{k - |S_v|}$ and $|S_v| < k$ **then**
            $S_v := S_v \cup \{e_i\}$
return $\text{argmax}_{v \in O_n} f(S_v)$

THEOREM 5.3. SIEVE-STREAMING *(Algorithm 3) satisfies the following properties*

- *It outputs a set S such that $|S| \leq k$ and $f(S) \geq \left(\frac{1}{2} - \epsilon\right)$ OPT*
- *It does 1 pass over the data set, stores at most $O\left(\frac{k \log k}{\epsilon}\right)$ elements and has $O\left(\frac{\log k}{\epsilon}\right)$ update time per element.*

The Sieve-streaming algorithm

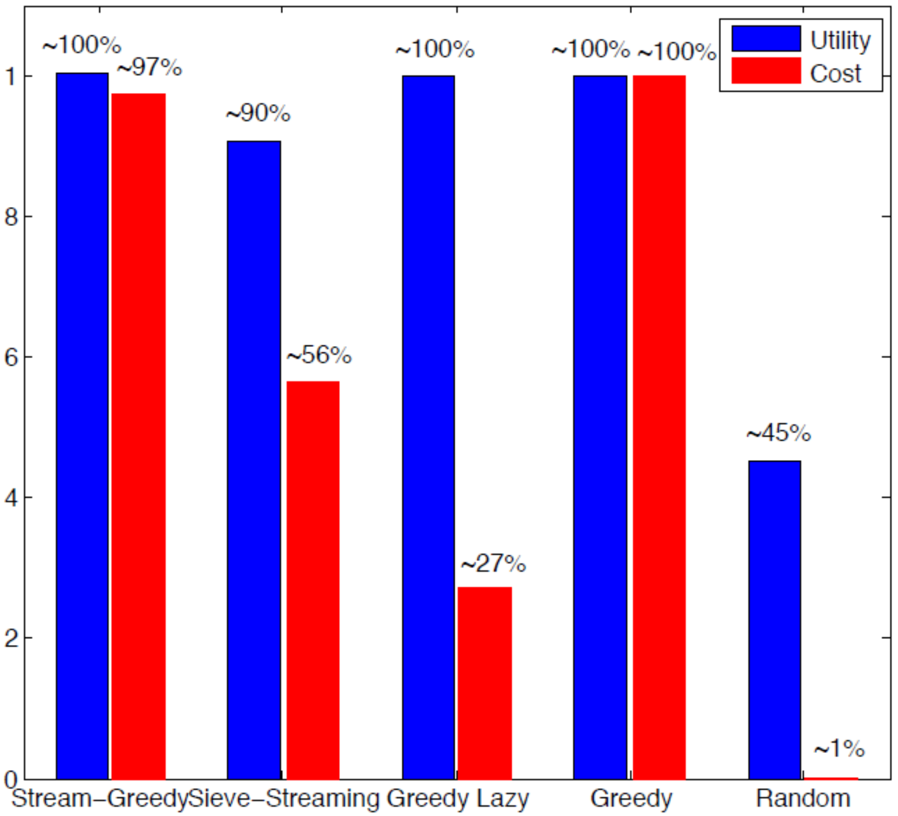


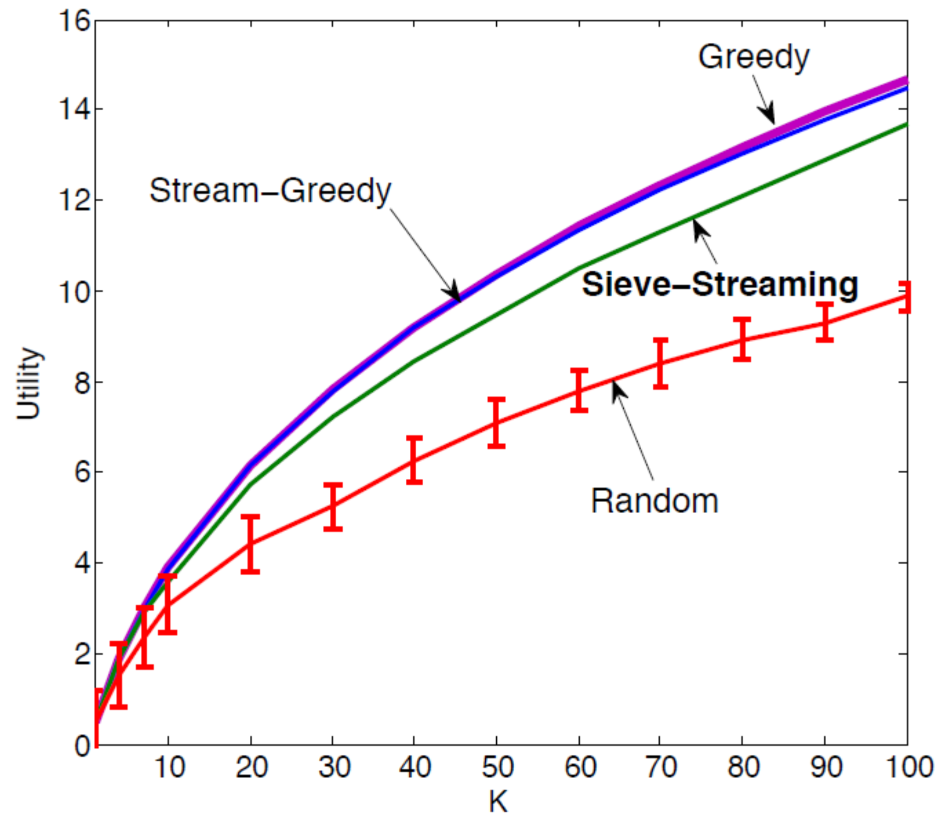
Experimental results

- Two applications:
  - Exemplar-based clustering
  - Active set selection for nonparametric learning
- Baselines vs. the Sieve-streaming algorithm
  - Random selection
  - Standard greedy
  - Lazy greedy
  - Stream greedy

Many data summarization

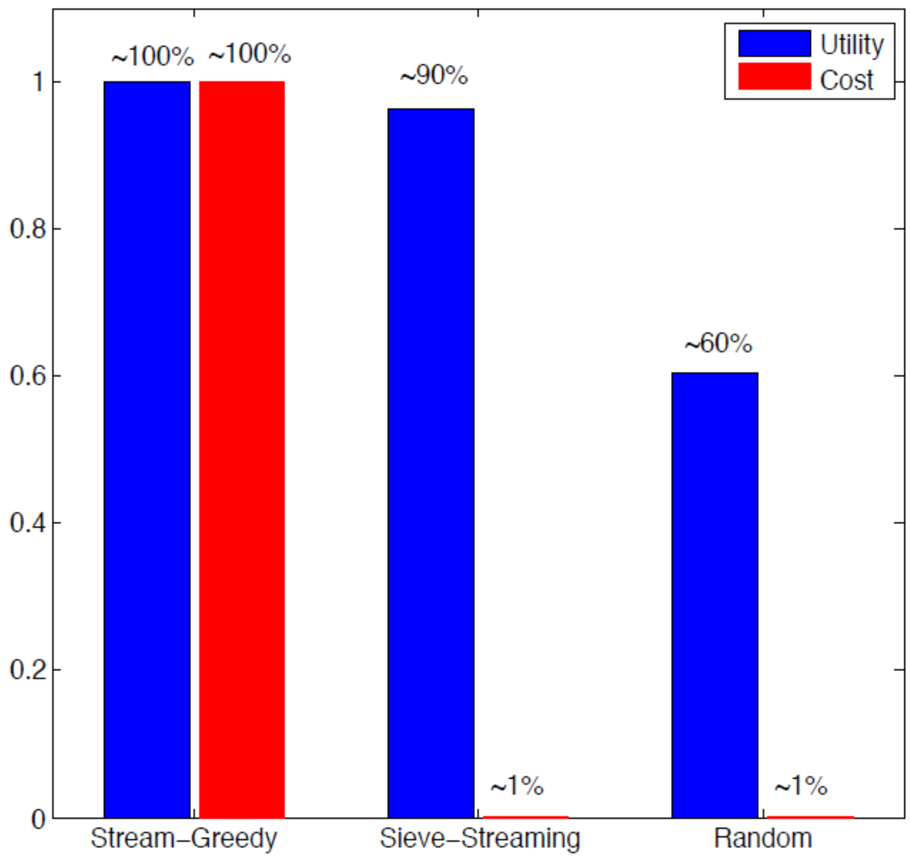
- Active set selection (5875 inst. 22 feats)

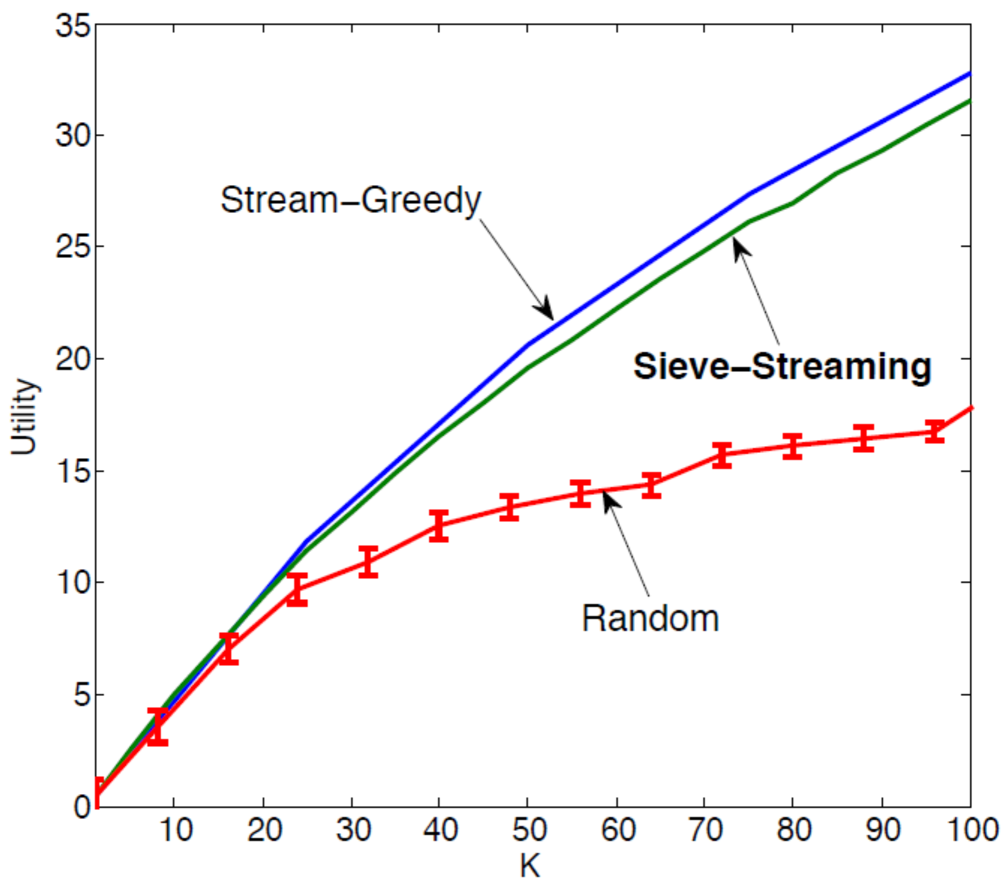




Many data summarization

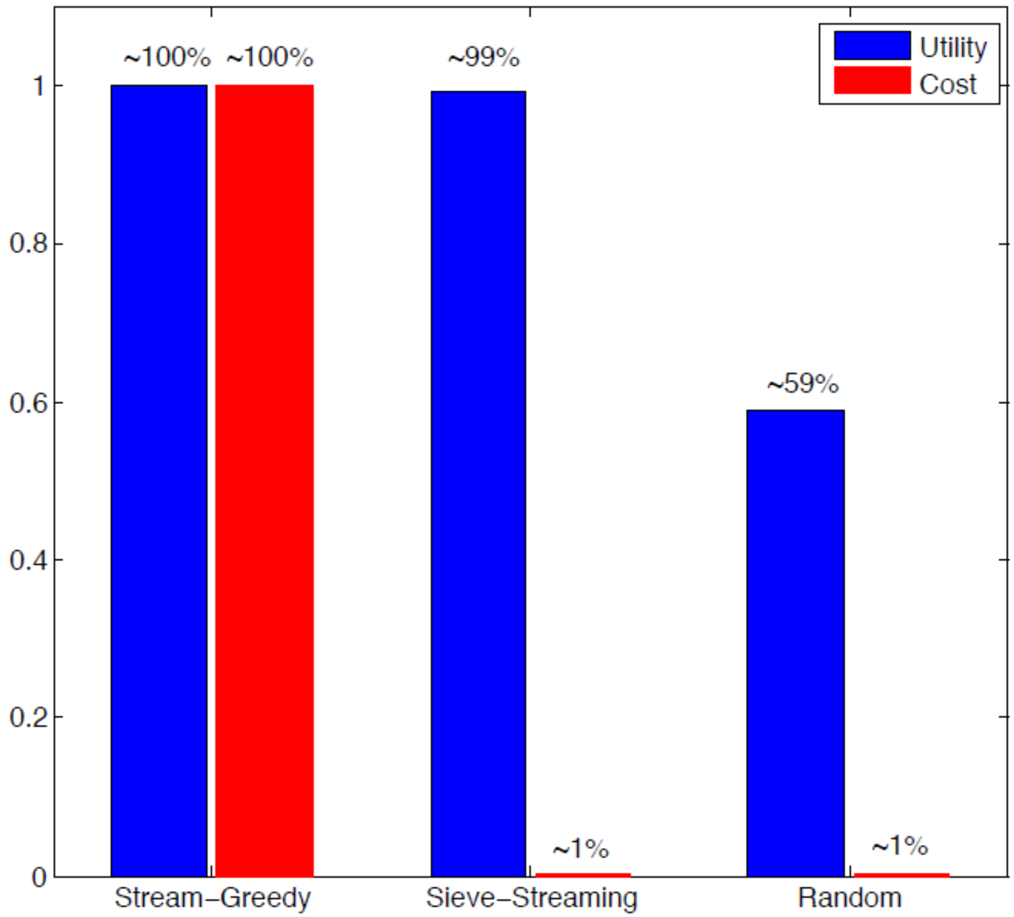
- Active set selection (+45 million inst. 6 feats)

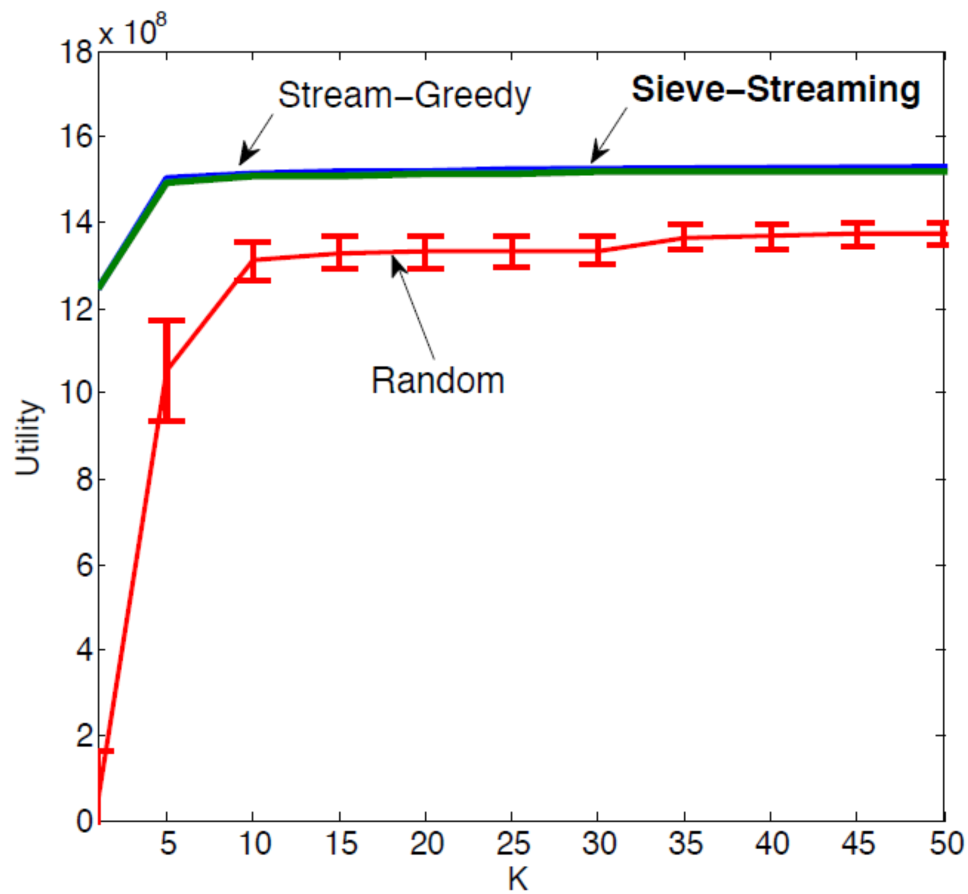




Many data summarization

- Exemplar-based clustering (2.5 million inst. 68 feats.)





Discussion

- Nice algorithms:
  - Easy to implement
  - Performance guarantees
  - Too many applications
- But, still have to be evaluated in the corresponding tasks
- Main challenge on using SFM: proof your objective function is monotone submodular
- How good is the guarantee for different tasks?
- There is already a lot of (ongoing) work on the use of SFM for diverse tasks
- **Matlab Toolbox for Submodular Function Optimization (v 2.0)**

http://las.ethz.ch/sfo/

Final remarks

- Research opportunities sith SFM:
  - NMF on a budget with SFM: what criterion?
  - Prototype selection/generation for NN classification (instance selection)
  - Vocabulary learning/construction for BoVWs: replace k-means with SFM of a supervised criterion
  - Multimodal document summarization / multimodal snippet generation: define appropriate criteria for SFM

References

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