

Homework No. 2

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Course: Fundamentals of Digital Signal Processing

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1 Problems

1. Given the following unit sample response sequences compute the Fourier transforms and sketch the magnitude and phase functions:

- (a) $h(0) = 0.5$; $h(-1) = h(1) = 1$; $h(n) = 0$ otherwise
(b) $h(0) = 1$; $h(1) = -1$; $h(n) = 0$ otherwise
(c) $h(0) = 1$; $h(1) = -0.5$; $h(2) = 1$; $h(n) = 0$ otherwise

Solution:

- (a) Lets compute the filter's transfer function as follows

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} \\ &= \sum_{n=-1}^1 h(n)e^{-j\omega n} \\ &= e^{j\omega} + \frac{1}{2} + e^{-j\omega} \\ &= \frac{1}{2} + 2\cos(\omega) \\ &= \frac{1 + 4\cos(\omega)}{2} \end{aligned}$$

Therefore, sequence's Fourier transform is a real-valued function of ω and takes both positive and negative values, as a consequence, we must add π to the resulting phase response whenever the magnitude response is negative. The expressions for magnitude response and phase response are as follows

$$|H(e^{j\omega})| = \left| \frac{1 + 4\cos(\omega)}{2} \right|$$

$$\text{Arg}[H(e^{j\omega})] = \begin{cases} 0 & H(e^{j\omega}) > 0 & -\arccos(-\frac{1}{4}) < \omega < \arccos(-\frac{1}{4}) \\ \pm\pi & H(e^{j\omega}) < 0 & -\pi \leq \omega < -\arccos(-\frac{1}{4}) \text{ or } \arccos(-\frac{1}{4}) < \omega < \pi \end{cases}$$

We must express transfer function as a rational function in order to determine the coefficients of both its numerator and denominator, which are needed to plot magnitude response and

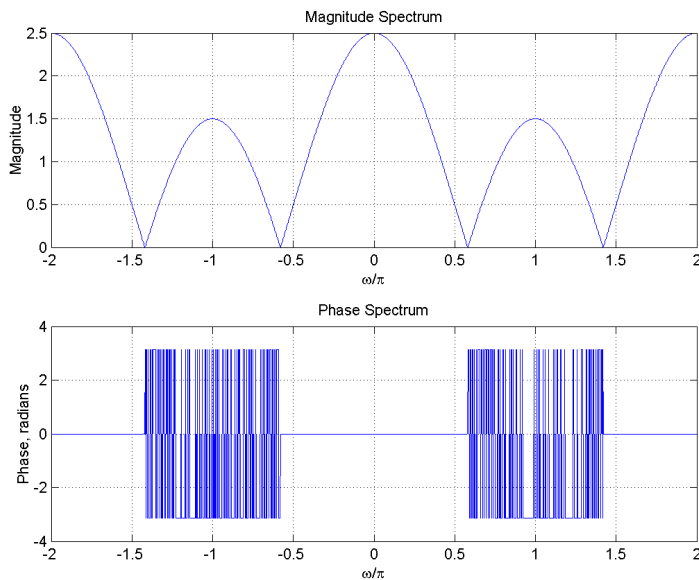


Figure 1: Magnitude response and phase response for sequence (a) in problem 1.1.

phase response in MATLAB. This is done below

$$H(e^{j\omega}) = \frac{2 + e^{-j\omega} + 2e^{-2j\omega}}{2e^{-j\omega}}$$

Figure 1 depicts plots of both magnitude response and phase response.

(b) Lets compute the filter's transfer function as follows

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} \\ &= \sum_{n=0}^1 h(n)e^{-j\omega n} \\ &= 1 - e^{-j\omega} \\ &= 1 - (\cos(-\omega) + j\sin(-\omega)) \\ &= 1 - (\cos(\omega) - j\sin(\omega)) \\ &= (1 - \cos(\omega)) + j\sin(\omega) \end{aligned}$$

This is a complex-valued function expressed as a sum of a real part and an imaginary part. Next we use the definitions of the magnitude response in terms of the transfer function and its complex conjugate and the phase response in terms of transfer function's real and imaginary parts

$$\begin{aligned} |H(e^{j\omega})| &= (H(e^{j\omega})H^*(e^{j\omega}))^{\frac{1}{2}} \\ &= ((1 - e^{-j\omega})(1 - e^{j\omega}))^{\frac{1}{2}} \\ &= (1 - e^{-j\omega} - e^{j\omega} + 1)^{\frac{1}{2}} \end{aligned}$$

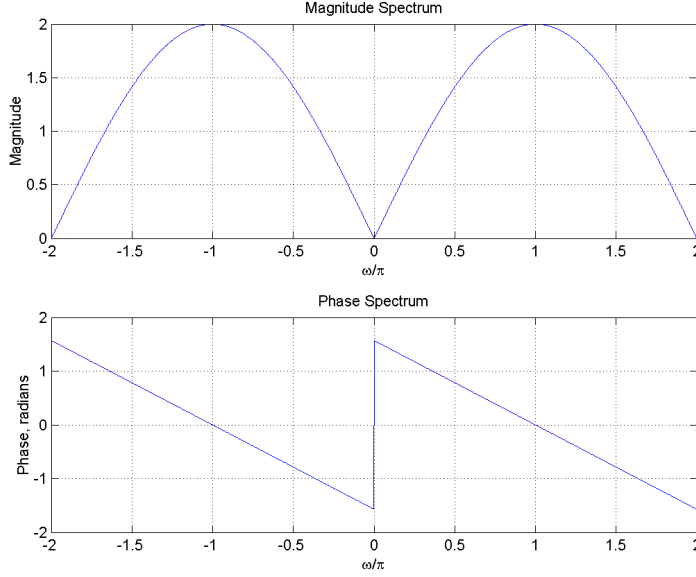


Figure 2: Magnitude response and phase response for sequence (b) in problem 1.1.

$$\begin{aligned}
 &= (2 - (e^{j\omega} + e^{-j\omega}))^{\frac{1}{2}} \\
 &= (2 - 2\cos(\omega))^{\frac{1}{2}} \\
 &= (2(1 - \cos(\omega)))^{\frac{1}{2}}
 \end{aligned}$$

$$\text{Arg}[H(e^{j\omega})] = \arctan\left(\frac{\sin(\omega)}{1 - \cos(\omega)}\right)$$

In this case, during the computation of the transfer function we got its rational function in an intermediate step, so we got the coefficients for plotting. Figure 2 illustrates the plots for these responses.

(c) Lets compute the filter's transfer function as follows

$$\begin{aligned}
 H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} \\
 &= \sum_{n=0}^2 h(n)e^{-j\omega n} \\
 &= 1 - \frac{1}{2}e^{-j\omega} + e^{-2j\omega} \\
 &= 1 - \frac{1}{2}(\cos(-\omega) + j\sin(-\omega)) + (\cos(-\omega) + j\sin(-\omega))^2 \\
 &= 1 - \frac{1}{2}(\cos(\omega) - j\sin(\omega)) + (\cos(\omega) - j\sin(\omega))^2 \\
 &= 1 - \frac{1}{2}\cos(\omega) + \frac{1}{2}j\sin(\omega) + \cos^2(\omega) - 2j\cos(\omega)\sin(\omega) - \sin^2(\omega) \\
 &= 1 - \sin^2(\omega) + \cos^2(\omega) - \frac{1}{2}\cos(\omega) + \frac{1}{2}j\sin(\omega) - 2j\cos(\omega)\sin(\omega)
 \end{aligned}$$

$$\begin{aligned}
&= \cos^2(\omega) + \cos^2(\omega) - \frac{1}{2}\cos(\omega) + j\sin(\omega)\left(\frac{1}{2} - 2\cos(\omega)\right) \\
&= 2\cos^2(\omega) - \frac{1}{2}\cos(\omega) + j\sin(\omega)\left(\frac{1}{2} - 2\cos(\omega)\right) \\
&= -\cos(\omega)\left(\frac{1}{2} - 2\cos(\omega)\right) + j\sin(\omega)\left(\frac{1}{2} - 2\cos(\omega)\right)
\end{aligned}$$

As we can see, what we have obtained is a complex-valued function of ω . Now we will use the definitions to compute the magnitude response in terms of the transfer function and its complex conjugate, as well as the phase response in terms of the transfer function's real and imaginary parts.

$$\begin{aligned}
|H(e^{j\omega})| &= (H(e^{j\omega})H^*(e^{j\omega}))^{\frac{1}{2}} \\
&= \left[\left(-\cos(\omega)\left(\frac{1}{2} - 2\cos(\omega)\right) + j\sin(\omega)\left(\frac{1}{2} - 2\cos(\omega)\right) \right) \right. \\
&\quad \left. \left(-\cos(\omega)\left(\frac{1}{2} - 2\cos(\omega)\right) - j\sin(\omega)\left(\frac{1}{2} - 2\cos(\omega)\right) \right) \right]^{\frac{1}{2}} \\
&= \left[\left(\cos(\omega)\left(\frac{1}{2} - 2\cos(\omega)\right) \right)^2 + \left(\sin(\omega)\left(\frac{1}{2} - 2\cos(\omega)\right) \right)^2 \right]^{\frac{1}{2}} \\
&= \left[\cos^2(\omega)\left(\frac{1}{2} - 2\cos(\omega)\right)^2 + \sin^2(\omega)\left(\frac{1}{2} - 2\cos(\omega)\right)^2 \right]^{\frac{1}{2}} \\
&= \left[\left(\frac{1}{2} - 2\cos(\omega)\right)^2 (\cos^2(\omega) + \sin^2(\omega)) \right]^{\frac{1}{2}} \\
&= \left[\left(\frac{1}{2} - 2\cos(\omega)\right)^2 \right]^{\frac{1}{2}} \\
&= \left| \frac{1}{2} - 2\cos(\omega) \right|
\end{aligned}$$

$$\begin{aligned}
\text{Arg}[H(e^{j\omega})] &= \arctan\left(-\frac{\sin(\omega)\left(\frac{1}{2} - 2\cos(\omega)\right)}{\cos(\omega)\left(\frac{1}{2} - 2\cos(\omega)\right)} \right) \\
&= \arctan\left(-\frac{\sin(\omega)}{\cos(\omega)} \right) \\
&= -\arctan(\tan(\omega)) \\
&= -\omega
\end{aligned}$$

At first sight we have got a linear function of ω as phase response. However lets compute the linear phase form of the transfer function

$$\begin{aligned}
H(e^{j\omega}) &= -\frac{1}{2}e^{-j\omega} + e^{-j\omega}(e^{j\omega} + e^{-j\omega}) \\
&= -\frac{1}{2}e^{-j\omega} + 2\cos(\omega)e^{-j\omega} \\
&= \underbrace{\left(2\cos(\omega) - \frac{1}{2}\right)}_{\text{amplitude function}} e^{-j\omega}
\end{aligned}$$

The amplitude function takes both positive and negative values, so we must take into account to add π to the earlier linear phase. The correct expression for the phase response for the filter

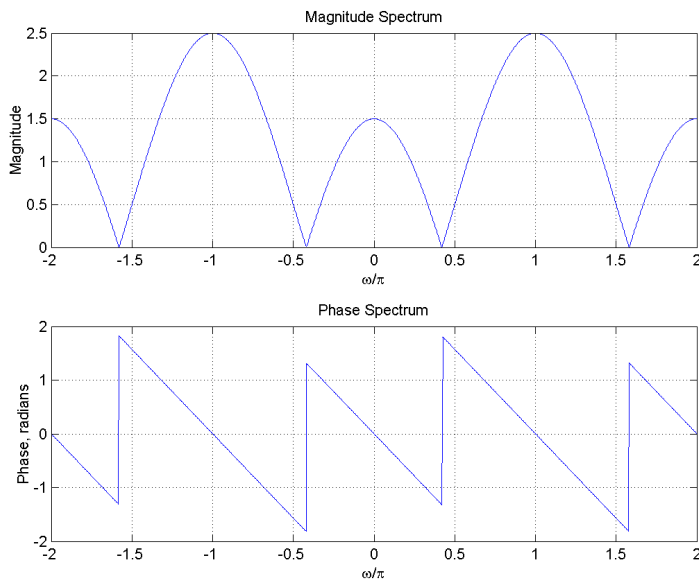


Figure 3: Magnitude response and phase response for sequence (c) in problem 1.1.

is as follows

$$\text{Arg}[H(e^{j\omega})] = \begin{cases} -\omega & -\arccos(\frac{1}{4}) < \omega < \arccos(\frac{1}{4}) \\ -\omega \pm \pi & \arccos(\frac{1}{4}) < \omega < \pi \text{ or } -\pi < \omega < -\arccos(\frac{1}{4}) \end{cases}$$

Finally, the plots for magnitude response and phase response are shown in figure 3.

- Determine and plot magnitude and phase responses of the first order recursive filter whose unit sample response is given by:

$$h(n) = a^n u(n) \quad \forall n \in \mathbb{Z}$$

Consider the value $a = 0.2$

Solution: Lets compute the filter's transfer function as follows

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} a^n u(n)e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} a^n e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} (ae^{-j\omega})^n \\ &= \frac{1}{1 - ae^{-j\omega}} \quad \text{since } |a| = |0.2| < 1 \\ &= \frac{1}{1 - a(\cos(-\omega) + j\sin(-\omega))} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{1 - a(\cos(\omega) - j\sin(\omega))} \\
&= \frac{1}{(1 - a\cos(\omega)) + aj\sin(\omega)} \\
&= \frac{1}{(1 - a\cos(\omega)) + aj\sin(\omega)} \frac{(1 - a\cos(\omega)) - aj\sin(\omega)}{(1 - a\cos(\omega)) - aj\sin(\omega)} \\
&= \frac{(1 - a\cos(\omega)) - aj\sin(\omega)}{(1 - a\cos(\omega))^2 + a^2\sin^2(\omega)} \\
&= \frac{(1 - a\cos(\omega)) - aj\sin(\omega)}{1 - 2a\cos(\omega) + a^2\cos^2(\omega) + a^2\sin^2(\omega)} \\
&= \frac{(1 - a\cos(\omega)) - aj\sin(\omega)}{1 - 2a\cos(\omega) + a^2} \\
&= \frac{1 - a\cos(\omega)}{1 - 2a\cos(\omega) + a^2} - j \frac{a\sin(\omega)}{1 - 2a\cos(\omega) + a^2}
\end{aligned}$$

As we can see, what we have obtained is a complex-valued function of ω . Now we will use the definitions to compute the magnitude response and the phase response in terms of the transfer function's real and imaginary parts.

$$\begin{aligned}
|H(e^{j\omega})| &= \left[\left(\frac{1 - a\cos(\omega)}{1 - 2a\cos(\omega) + a^2} \right)^2 + \left(- \frac{a\sin(\omega)}{1 - 2a\cos(\omega) + a^2} \right)^2 \right]^{\frac{1}{2}} \\
&= \left[\frac{(1 - a\cos(\omega))^2}{(1 - 2a\cos(\omega) + a^2)^2} + \frac{a^2\sin^2(\omega)}{(1 - 2a\cos(\omega) + a^2)^2} \right]^{\frac{1}{2}} \\
&= \left[\frac{1 - 2a\cos(\omega) + a^2\cos^2(\omega) + a^2\sin^2(\omega)}{(1 - 2a\cos(\omega) + a^2)^2} \right]^{\frac{1}{2}} \\
&= \left[\frac{1 - 2a\cos(\omega) + a^2}{(1 - 2a\cos(\omega) + a^2)^2} \right]^{\frac{1}{2}} \\
&= \left[\frac{1}{1 - 2a\cos(\omega) + a^2} \right]^{\frac{1}{2}} \\
&= \left(\frac{1}{1.04 - 0.4\cos(\omega)} \right)^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
\text{Arg}[H(e^{j\omega})] &= \arctan \left(- \frac{a\sin(\omega)}{1 - 2a\cos(\omega) + a^2} \bigg/ \frac{1 - a\cos(\omega)}{1 - 2a\cos(\omega) + a^2} \right) \\
&= \arctan \left(- \frac{a\sin(\omega)}{1 - a\cos(\omega)} \right) \\
&= -\arctan \left(\frac{a\sin(\omega)}{1 - a\cos(\omega)} \right) \\
&= -\arctan \left(\frac{0.2\sin(\omega)}{1 - 0.2\cos(\omega)} \right)
\end{aligned}$$

Finally, the plots for magnitude response and phase response are depicted in figure 4.

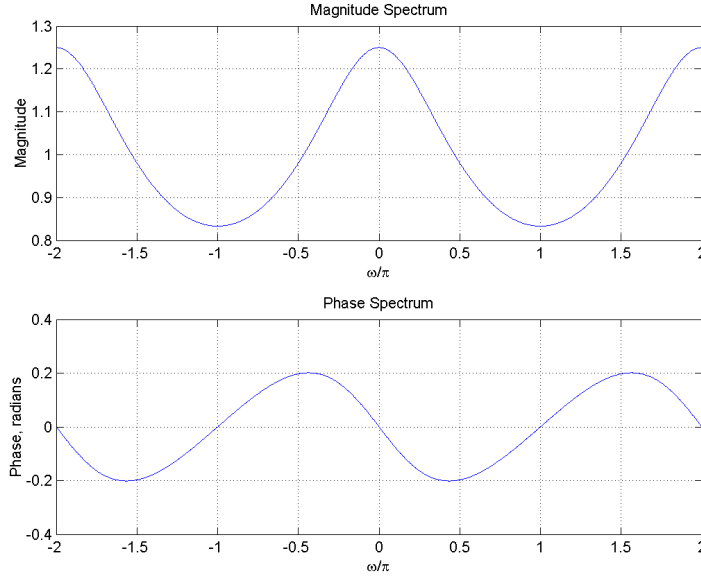


Figure 4: Magnitude response and phase response for the first order recursive filter.

- Determine the convolution of the sequences $x_1(n)$ and $x_2(n)$ in the frequency domain by using their Fourier transforms.

$$x_1(n) = x_2(n) = \begin{cases} 1 & n = 0 \\ 1 & n = 1 \\ 1 & n = 2 \\ 0 & \text{elsewhere} \end{cases}$$

Solution: In order to determine the convolution of the sequences we must compute their Fourier transforms, which in this case turns out to be the same for both sequences. The procedure is the following

$$\begin{aligned} X_1(e^{j\omega}) &= X_2(e^{j\omega}) \\ &= \sum_{n=-\infty}^{\infty} x_1(n)e^{-j\omega n} \\ &= \sum_{n=0}^2 x_1(n)e^{-j\omega n} \\ &= 1 + e^{-j\omega} + e^{-2j\omega} \\ &= 1 + (\cos(-\omega) + j\sin(-\omega)) + (\cos(-\omega) + j\sin(-\omega))^2 \\ &= 1 + (\cos(\omega) - j\sin(\omega)) + (\cos(\omega) - j\sin(\omega))^2 \\ &= 1 + \cos(\omega) - j\sin(\omega) + \cos^2(\omega) - 2j\cos(\omega)\sin(\omega) - \sin^2(\omega) \\ &= 1 - \sin^2(\omega) + \cos^2(\omega) + \cos(\omega) - j\sin(\omega) - 2j\cos(\omega)\sin(\omega) \\ &= \cos(\omega) - j\sin(\omega)(1 + 2\cos(\omega)) \end{aligned}$$

Once we have computed Fourier transform for both sequences we can use it to obtain the convolution

in frequency domain of the original sequences. Convolution, therefore, is given by

$$\begin{aligned} X_1(e^{j\omega})X_2(e^{j\omega}) &= \left(\cos(\omega) - j\sin(\omega)(1 + 2\cos(\omega)) \right)^2 \\ &= \cos^2(\omega) - 2j\cos(\omega)\sin(\omega)(1 + 2\cos(\omega)) - \sin^2(\omega)(1 + 2\cos(\omega))^2 \\ &= \cos^2(\omega) - 2j\cos(\omega)\sin(\omega)(1 + 2\cos(\omega)) - \sin^2(\omega)(1 + 4\cos(\omega) + 4\cos^2(\omega)) \end{aligned}$$

2 Computer Projects

1. Find magnitude and phase responses for the filter

$$h(0) = 1 = h(7); h(1) = 2 = h(6); h(2) = 3 = h(5); h(3) = 4 = h(4)$$

Solution: This is an even-symmetric sequence whose transfer function is the following polynomial in $e^{-j\omega}$. The coefficients of the polynomial are the values of $h(n)$ for every n .

$$H(e^{j\omega}) = 1 + 2e^{-j\omega} + 3e^{-2j\omega} + 4e^{-3j\omega} + 4e^{-4j\omega} + 3e^{-5j\omega} + 2e^{-6j\omega} + e^{-7j\omega}$$

We use the coefficients of the filter's transfer function in the following code which uses the `freqz` MATLAB function to compute and plot magnitude response and phase response for all frequencies in the interval $[-2\pi, 2\pi]$

```
w = -2*pi:0.01:2*pi;

% Coefficients for problem 2.1
num = [1 2 3 4 4 3 2 1];
den = [1];

h = freqz(num, den, w);

subplot(2, 1, 1);
plot(w/pi, abs(h)); grid
title('Magnitude Spectrum')
xlabel('\omega/\pi'); ylabel('Magnitude')

subplot(2, 1, 2);
plot(w/pi, angle(h)); grid
title('Phase Spectrum')
xlabel('\omega/\pi'); ylabel('Phase, radians')
```

The plots produced by this program are depicted in figure 5.

2. For the following difference equation determine magnitude response and phase response of the filter

$$y(n) = x(n) - x(n - N), \quad N = 5$$

Solution: In this case we must first determine the values of unit sample response in order to find the coefficients which are given to the `freqz` function. If we set $\{x(n)\} = \{\delta(n)\}$ we will only get nonzero values of $y(n)$ in $n = 0$ and $n = 5$. The unit sample response is therefore

$$h(n) = \begin{cases} 1 & n = 0 \\ -1 & n = 5 \\ 0 & \text{otherwise} \end{cases}$$

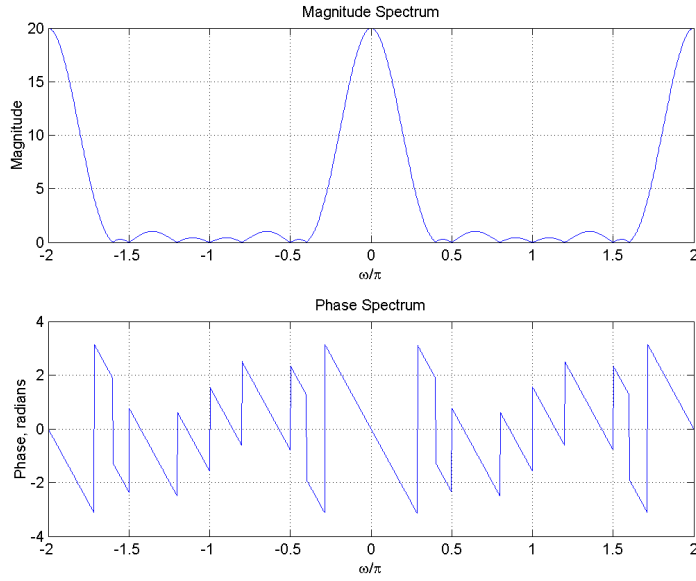


Figure 5: Magnitude response and phase response for the filter in problem 2.1.

The transfer function corresponding to the difference equation above turns out to be

$$H(e^{j\omega}) = 1 - e^{-5j\omega} = (1 - \cos(5\omega)) + j\sin(5\omega)$$

Magnitude response and phase response are as follows

$$|H(e^{j\omega})| = \left(2(1 - \cos(5\omega))\right)^{\frac{1}{2}}$$

$$\text{Arg}[H(e^{j\omega})] = \arctan\left(\frac{\sin(5\omega)}{1 - \cos(5\omega)}\right)$$

The following code in MATLAB was used to get the plots of magnitude response and phase response which are depicted in figure 6

```
w = -2*pi:0.01:2*pi;

% Coefficients for problem 2.2
num = [1 0 0 0 0 -1];
den = [1];

h = freqz(num, den, w);

subplot(2, 1, 1);
plot(w/pi, abs(h)); grid
title('Magnitude Spectrum')
xlabel('\omega/\pi'); ylabel('Magnitude')

subplot(2, 1, 2);
```

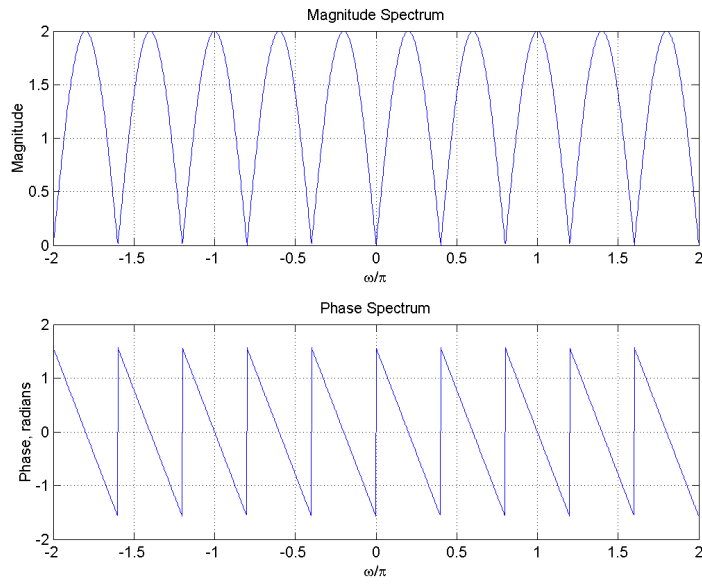


Figure 6: Magnitude response and phase response for the difference equation in problem 2.2.

```
plot(w/pi, angle(h)); grid
title('Phase Spectrum')
xlabel('\omega/\pi'); ylabel('Phase, radians')
```