Homework No. 1

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Course: Digital Signal Processing

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1 Problems

1. Find unit-sample response $h(n,0)$ and $h(n,1)$ for the first order recursive filter:

$$y(n) = \begin{cases} ay(n-1) + x(n) & \text{for } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Solution: In order to find $h(n,0)$ let $x(n) = \delta(n)$. So we get the following values for $y(n)$ in terms of n:

$$\begin{aligned}
y(0) &= ay(-1) + \delta(0) \\
&= a \cdot 0 + 1 \qquad \text{since } y(0) = 0 \;\; \forall\, n < 0 \\
&= 1 \\
y(1) &= ay(0) + \delta(1) \\
&= a \cdot 1 + 0 \\
&= a \\
y(2) &= ay(1) + \delta(2) \\
&= a \cdot a + 0 \\
&= a^2 \\
y(3) &= ay(2) + \delta(3) \\
&= a \cdot a^2 + 0 \\
&= a^3 \\
&\vdots \\
y(k) &= ay(k-1) + \delta(k) \\
&= a \cdot a^{k-1} + 0 \\
&= a^k
\end{aligned}$$

Therefore we can conclude that $h(n,0) = a^n u(n) \;\; \forall\, n \in \mathbb{Z}$. If $0 < a < 1$ then $y(n) = h(n,0)$ is an strictly decreasing sequence.

In order to find $h(n,1)$ let $x(n) = \delta(n-1)$. So we get the following values for $y(n)$ in terms of n:

$$\begin{aligned}
y(0) &= ay(-1) + \delta(-1) \\
&= a \cdot 0 + 0 \\
&= 0 \\
y(1) &= ay(0) + \delta(0) \\
&= a \cdot 0 + 1 \\
&= 1 \\
y(2) &= ay(1) + \delta(1) \\
&= a \cdot 1 + 0 \\
&= a \\
y(3) &= ay(2) + \delta(2) \\
&= a \cdot a + 0 \\
&= a^2 \\
&\vdots \\
y(k) &= ay(k-1) + \delta(k-1) \\
&= a \cdot a^{k-1} + 0 \\
&= a^k
\end{aligned}$$

Therefore we can conclude that $h(n,1) = a^{n-1}u(n-1) \;\; \forall\, n \in \mathbb{Z}$. Which is an strictly decreasing sequence as well.

2. Given the following sequences:

$$x(n) = \begin{cases} n+1 & 0 \leq n \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$h(n) = a^n u(n) \quad \text{for all } n, a = 0.98$$

Determine and illustrate the convolution.

Solution: The convolution of these sequences is computed by the following expression:

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

Lets calculate the values of y in terms of n:

$$\begin{aligned}
y(0) &= \sum_{k=-\infty}^{\infty} x(k)h(-k) \\
&= x(0)h(0)
\end{aligned}$$

$$
\begin{aligned}
&= 1 \cdot 1 \\
&= 1 \\
y(1) &= \sum_{k=-\infty}^{\infty} x(k)h(1-k) \\
&= x(0)h(1) + x(1)h(0) \\
&= 1 \cdot 0.98 + 2 \cdot 1 \\
&= 2.98 \\
y(2) &= \sum_{k=-\infty}^{\infty} x(k)h(2-k) \\
&= x(0)h(2) + x(1)h(1) + x(2)h(0) \\
&= 1 \cdot 0.98^2 + 2 \cdot 0.98 + 3 \cdot 1 \\
&= 5.9204 \\
&\vdots \\
y(n) &= \sum_{k=-\infty}^{\infty} x(k)h(n-k) \\
&= 0.98^n u(n) + 2 \cdot 0.98^{n-1} u(n-1) + 3 \cdot 0.98^{n-2} u(n-2)
\end{aligned}
$$

Therefore

$$x(n) * h(n) = 0.98^n u(n) + 2 \cdot 0.98^{n-1} u(n-1) + 3 \cdot 0.98^{n-2} u(n-2)$$

is the convolution sequence and is depicted in figure 6.

3. Test the stability of the first order recursive filter:

$$y(n) = ay(n-1) + x(n)$$

using zero initial conditions.

Solution: The unit sample response for this system is $h(n) = a^n u(n) \; \forall \; n \in \mathbb{Z}$. In order to test stability we must determine if

$$S = \sum_{k=-\infty}^{\infty} |h(k)| = \sum_{k=0}^{\infty} |a^n|$$

is finite. If $|a| \geq 1$ then S is increasing and unbounded, on the other hand if $|a| < 1$ then the expression above is equivalent to:

$$S = \frac{1}{1-a}, \quad |a| < 1$$

which is finite. Hence $y(n)$ is finite as long as $|a| < 1$.

4. For each of the following transformations, determine whether the system is *stable*, *causal*, *linear*, and *time invariant*.

a) $G_1\{x(n)\} = ax(n-n_0) + bx(n-n_1)$, a, b, n_0 and n_1 are constants

b) $G_2\{x(n)\} = x(n)x(n-n_0)$

Solution: First we test the properties for G_1. To test if the transformation is linear then let $x_1(n)$ and $x_2(n)$ be two different squences such that

$$y_1(n) = G_1\{x_1(n)\} = ax_1(n-n_0) + bx_1(n-n_1)$$

and

$$y_2(n) = G_1\{x_2(n)\} = ax_2(n-n_0) + bx_2(n-n_1)$$

then

$$\begin{aligned} G_1\{\alpha x_1(n) + \beta x_2(n)\} &= a\big[\alpha x_1(n-n_0) + \beta x_2(n-n_0)\big] + \\ &\quad b\big[\alpha x_1(n-n_1) + \beta x_2(n-n_1)\big] \\ &= a\alpha x_1(n-n_0) + b\alpha x_1(n-n_1) + \\ &\quad a\beta x_2(n-n_0) + b\beta x_2(n-n_1) \\ &= \alpha\big[ax_1(n-n_0) + bx_1(n-n_1)\big] + \\ &\quad \beta\big[ax_2(n-n_0) + bx_2(n-n_1)\big] \\ &= \alpha y_1(n) + \beta y_2(n) \end{aligned}$$

Therefore G_1 is a linear system.

We need to find $h(n)$ in order to determine if the system is stable, causual and time invariant. We have two cases, if $n_0 \neq n_1$ then

$$h(n) = \begin{cases} a & n = n_0 \\ b & n = n_1 \\ 0 & \text{otherwise} \end{cases}$$

and if $n_0 = n_1$ then

$$h(n) = \begin{cases} a+b & n = n_0 = n_1 \\ 0 & \text{otherwise} \end{cases}$$

To prove that the system is stable we must determine if

$$S = \sum_{k=-\infty}^{\infty} |h(k)|$$

is finite. In both cases $S = a + b$, where a, b are finite constants and thus S is finite too. Therefore G_1 is satable.

A system is causual if and only if $h(n) = 0 \; \forall \; n < 0$. In this case if $n = n_0$ or $n = n_1$ and if any of n_0 and n_1 was less than zero then $h(n)$ would be non-zero for $n < 0$. So in order for the system to be causual both n_0 and n_1 must be greater than or equal to zero.

To test time invariance let α be any constant and $y(n) = G_1\{x(n)\}$ then

$$G_1\{x(n-\alpha)\} = ax(n-\alpha-n_0) + bx(n-\alpha-n_1) = y(n-\alpha)$$

Therefore G_1 is time-invariant.

Now we will consider G_2 and test if it satisfies all of the criteria. Lets start with the superposition principle. Let $x_1(n)$ and $x_2(n)$ be any two sequences such that

$$y_1(n) = G_2\{x_1(n)\} = x_1(n)x_1(n-n_0)$$

and

$$y_2(n) = G_2\{x_2(n)\} = x_2(n)x_2(n-n_0)$$

then

$$\begin{aligned} G_2\{\alpha x_1(n) + \beta x_2(n)\} &= \big[\alpha x_1(n) + \beta x_2(n)\big]\big[\alpha x_1(n-n_0) + \beta x_2(n-n_0)\big] \\ &= \alpha^2 x_1(n)x_1(n-n_0) + \alpha\beta x_2(n)x_1(n-n_0) + \\ &\quad\ \alpha\beta x_1(n)x_2(n-n_0) + \beta^2 x_2(n)x_2(n-n_0) \\ &\neq \alpha x_1(n)x_1(n-n_0) + \beta x_2(n)x_2(n-n_0) \\ &= \alpha y_1(n) + \beta y_2(n) \end{aligned}$$

Therefore the transformation G_2 is not linear.

Now lets compute the unit sample response $h(n) = G_2\{\delta(n)\}$. Depending on the value of n_0 we have the following three cases:

a) $n_0 < 0$
If $n < n_0$ then $G_2\{\delta(n)\} = 0$
If $n = n_0$ then $G_2\{\delta(n)\} = \delta(n_0)\delta(0) = 0 \cdot 1 = 0$
If $n > n_0$ then $G_2\{\delta(n)\} = 0$
Hence $h(n) = 0 \ \forall \ n \in \mathbb{Z}$

b) $n_0 = 0$
If $n < 0$ then $G_2\{\delta(n)\} = 0$
If $n = 0$ then $G_2\{\delta(n)\} = \delta(0)\delta(0) = 1 \cdot 1 = 1$
If $n > 0$ then $G_2\{\delta(n)\} = 0$
Hence $h(n) = \delta(n) \ \forall \ n \in \mathbb{Z}$

c) $n_0 > 0$
If $n < n_0$ then $G_2\{\delta(n)\} = 0$
If $n = n_0$ then $G_2\{\delta(n)\} = \delta(n_0)\delta(0) = 0 \cdot 1 = 0$
If $n > n_0$ then $G_2\{\delta(n)\} = 0$
Hence $h(n) = 0 \ \forall \ n \in \mathbb{Z}$

Now we must test if the system is stable, i.e. if

$$S = \sum_{k=-\infty}^{\infty} |h(k)|$$

is finite. If $n_0 \neq 0$ then $S = 0$ and when $n_0 = 0$ then $S = 1$. In each case S is finite and therefore G_2 is stable no matter the value of n_0.

Since in every case $h(n) = 0 \ \forall \ n < 0$ then the system is causual as well.

Finally, lets test time-invariance. Let α be any constant, then

$$G_2\{x(n-\alpha)\} = x(n-\alpha)x(n-\alpha-n_0)$$

so G_2 is time-invariant.

5. For the following difference equations determine and sketch the unit sample response sequence and plot the digital filter structures:

   a) $y_1(n) = x(n) - x(n-N)$, $N = 4$
   b) $y_2(n) = ay(n-1) + x(n) + x(n-1)$, for $a < 1$

   Solution: Lets determine $h(n)$ for $y_1(n)$ using zero initial condition.

   If $n < 0$ then $\delta(n) = 0$ and $\delta(n-4) = 0$, so $y_1(n) = 0$

   If $n = 0$ then $y_1(0) = \delta(0) - \delta(-4) = 1 - 0 = 1$

   If $n = 1$ then $y_1(1) = \delta(1) - \delta(-3) = 0 - 0 = 0$

   If $n = 2$ then $y_1(2) = \delta(2) - \delta(-2) = 0 - 0 = 0$

   If $n = 3$ then $y_1(3) = \delta(3) - \delta(-1) = 0 - 0 = 0$

   If $n = 4$ then $y_1(4) = \delta(4) - \delta(0) = 0 - 1 = 1$

   If $n > 4$ then $\delta(n) = 0$ and $\delta(n-4) = 0$, so $y_1(n) = 0$

   Therefore we can conclude that

$$h(n) = \begin{cases} 1 & n = 0 \text{ or } n = 4 \\ 0 & \text{otherwise} \end{cases}$$

   Now lets determine $h(n)$ for $y_2(n)$ using zero initial conditions

   If $n < 0$ then $y_2(n) = 0$

   If $n = 0$ then $y_2(0) = a \cdot 0 + 1 + 0 = 1$

   If $n = 1$ then $y_2(1) = a \cdot 1 + 0 + 1 = a + 1$

   If $n = 2$ then $y_2(2) = a(a+1) + 0 + 0 = a^2 + a$

   If $n = 3$ then $y_2(3) = a(a^2 + a) + 0 + 0 = a^3 + a^2$

   If $n = 4$ then $y_2(4) = a(a^3 + a^2) + 0 + 0 = a^4 + a^3$

   $\vdots$

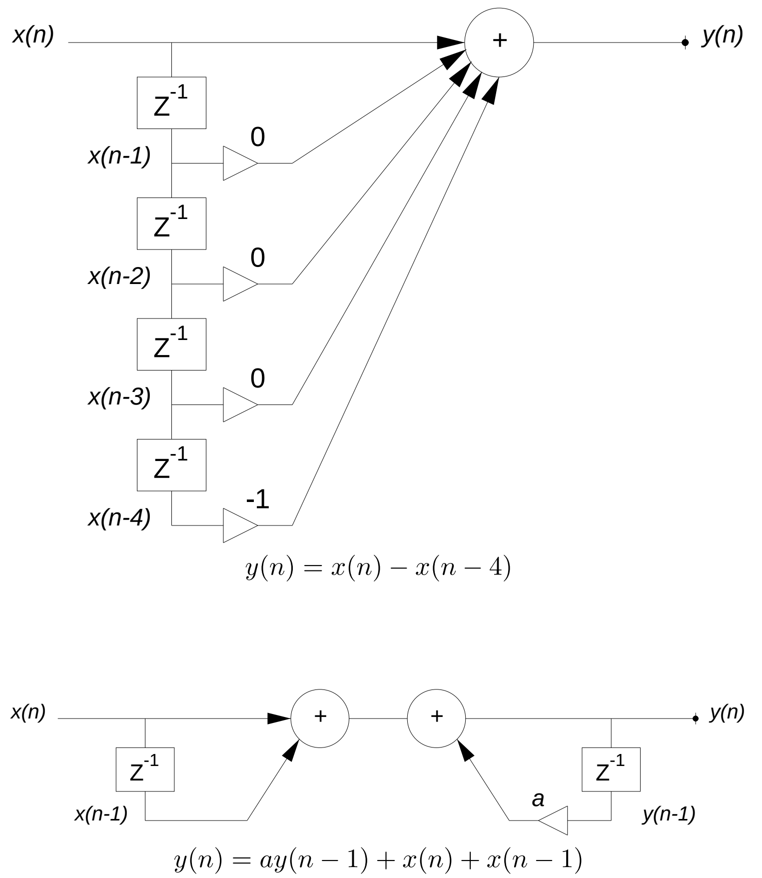


Figure 1: Digital filter structures.

If $n = k$ then $y_2(k) = a^k + a^{k+1}$

In this case we found that

$$h(n) = \begin{cases} a^n + a^{n+1} & n > 1 \\ 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

The digital filter structures corresponding to these difference equations are shown in figure 1.

2 Computer Projects

2.1 Discrete time sequences

1. Generate and plot unit sample sequence having $N = 30$ elements.

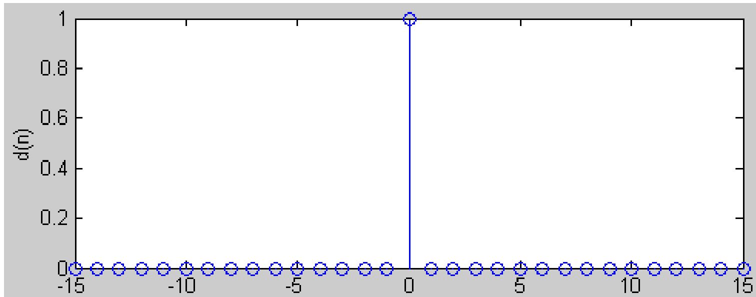


Figure 2: Unit sample sequence plotted in MATLAB.

Solution: The following piece of code is used to generate and plot unit sample sequence in MATLAB:

```
N = input('Length of unit sample sequence: ');
x = -N/2:1:N/2;
y = [zeros(1, N/2) 1 zeros(1, N/2)];
stem(x, y);
xlabel('n');
ylabel('d(n)');
```

And the resulting plot is depicted in figure 2.

2. Generate and plot unit step sequence having $N = 30$ elements.

   Solution: This time the code used to generate unit step sequence in MATLAB is the following:

```
N = input('Length of unit step sequence: ');
x = -N/2:1:N/2 - 1;
y = [zeros(1, N/2) ones(1, N/2)];
stem(x, y);
xlabel('n');
ylabel('u(n)');
```

   And the plot is shown in figure 3.

3. Generate and plot sinusoidal sequence having $N = 30$ elements and given value $\omega = \frac{\pi}{3}$.

   Solution: The code in MATLAB used to generate the sequence shown in figure 4 is the following:

```
N = input('Length of the sinusoidal sequence: ');
x = -N/2:1:N/2;
y = sin(pi/3 * x);
stem(x, y);
xlabel('n');
ylabel('sin(pi/3*n)');
```

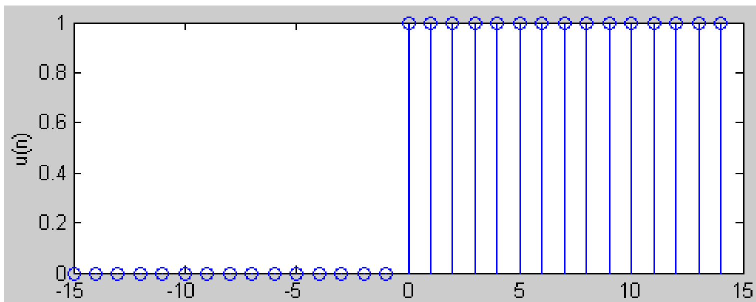


Figure 3: Unit step sequence plotted in MATLAB.

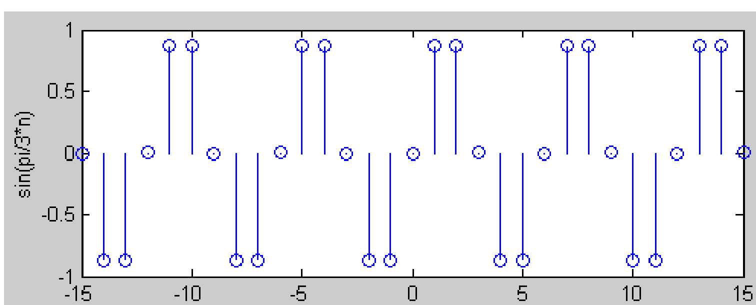
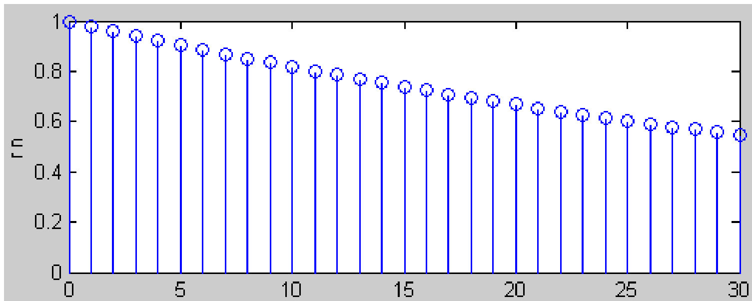
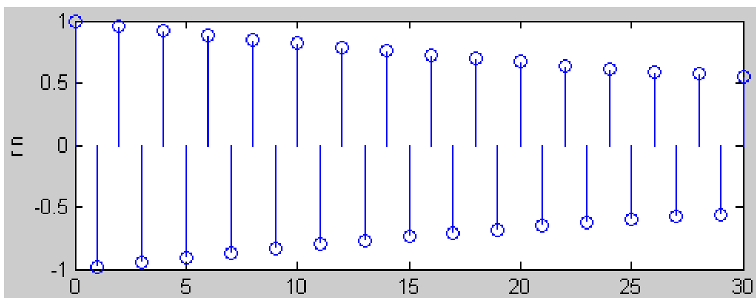


Figure 4: Sinusoidal sequence plotted in MATLAB.



(a) $r = 0.98$



(b) $r = -0.98$

Figure 5: Plot of the sequence $x(n) = r^n$ for different values of r.

4. Generate and plot real-valued sequence

$$x(n) = r^n$$

having $N = 30$ elements for $r = 0.98$. Repeat for same negative value $r = -0.98$.

Solution: Figure 5 shows the plots generated from the following code in MATLAB:

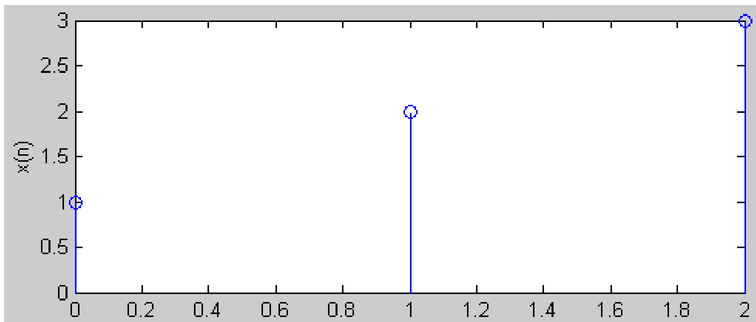
```
N = input('Length of the sequence: ');
r = 0.98; % altarnatively r = 0.98
x = 0:1:N;
y = r.^x;
stem(x,y);
xlabel('n');
ylabel('r^ n');
```

2.2 Convolution

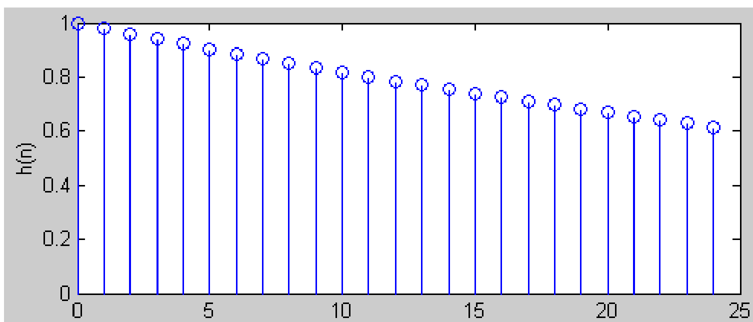
Compute and plot the convolution of the sequences given in problem 2, section 1.

Solution: Figure 6 shows the sequence $x(n)$, the unit sample response $h(n)$ and the result of convolution $x(n) * h(n)$. The code which computes and plots convolution for this example is the following:

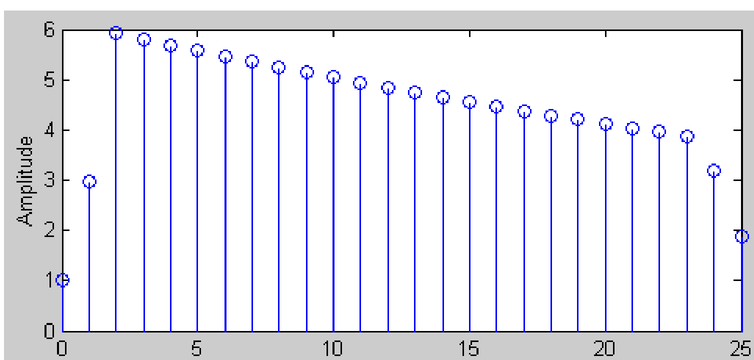
```
% Auxiliary sequences used to build h(n)
a = 0.98 * [ones(1, 24)];
b = [0:1:23];
% Builds the unit sample response sequence
h = a.^b;
% Builds the input sequence
x = [1 2 3];
% Computes convolution
c = conv(x, h);
n = length(c) - 1;
% Plots the convolution sequence
stem(0:1:n, c);
xlabel('n');
ylabel('Amplitude');
```



(a) $x(n)$



(b) $h(n)$



(c) $x(n) * h(n)$

Figure 6: An input sequence, unit sample response and convolution sequences.