Properties of Convolution

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Common Impulse Responses

Introduction

- Linear system's characteristics
  - System's impulse response
  - Convolution mathematics

- Examples:
  - Digital filters are created by designing an appropriate impulse response

  - Radar's detection system analyses an impulse response

  - Etc

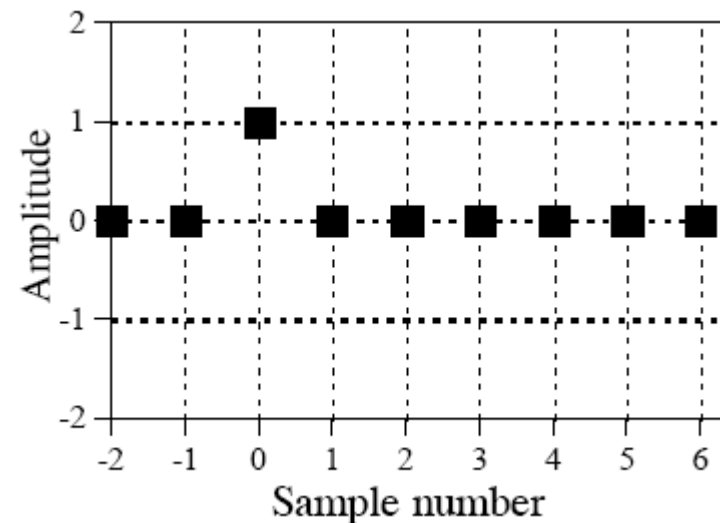# Common Impulse Responses

Delta Function

The simplest impulse response is nothing more than a delta function

An impulse on the input produces an identical impulse on the output

$$x[n] * \delta[n] = x[n]$$

This is the goal of systems that transmit or store signals.
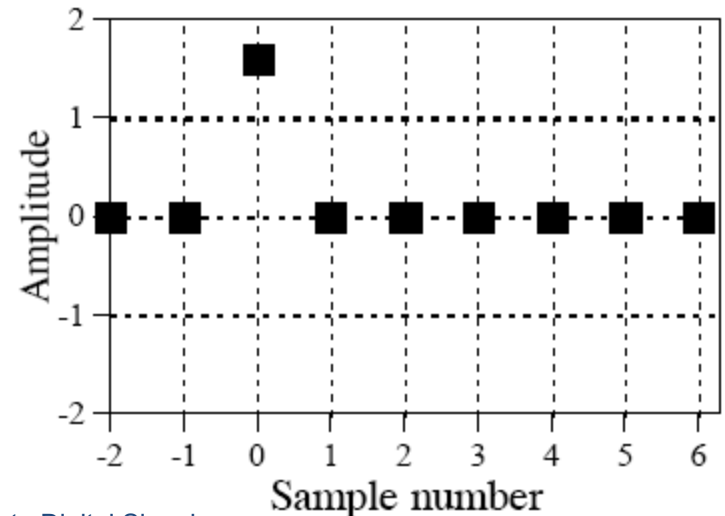

Convolution
Identity property

Common Impulse Responses

If the delta function is made larger or smaller in amplitude, the resulting system is an **amplifier or attenuator**, respectively.

In equation form, amplification results if k is greater than one, and attenuation results if k is less than one:

$$x[n] * k\delta[n] = kx[n]$$

This impulse response will amplify the signal by 1.6
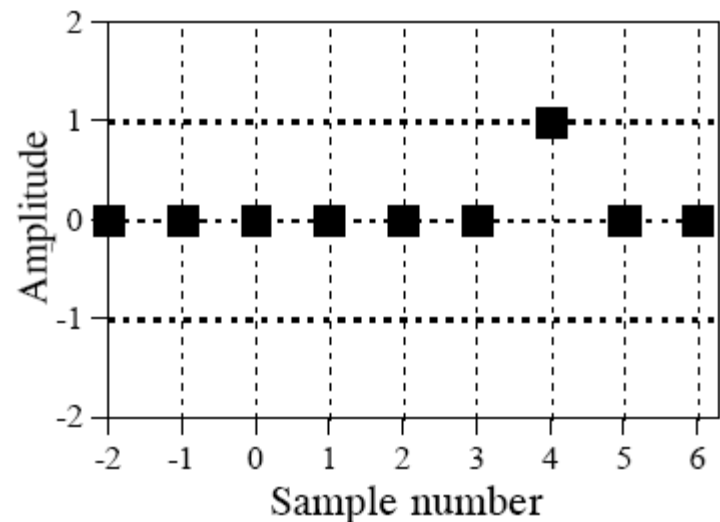
Common Impulse Responses

Delta function with a shift.
This results in a system that introduces an identical shift between the input and output signals.

Letting the shift be represented by the parameter, s, this can be written as the equation:
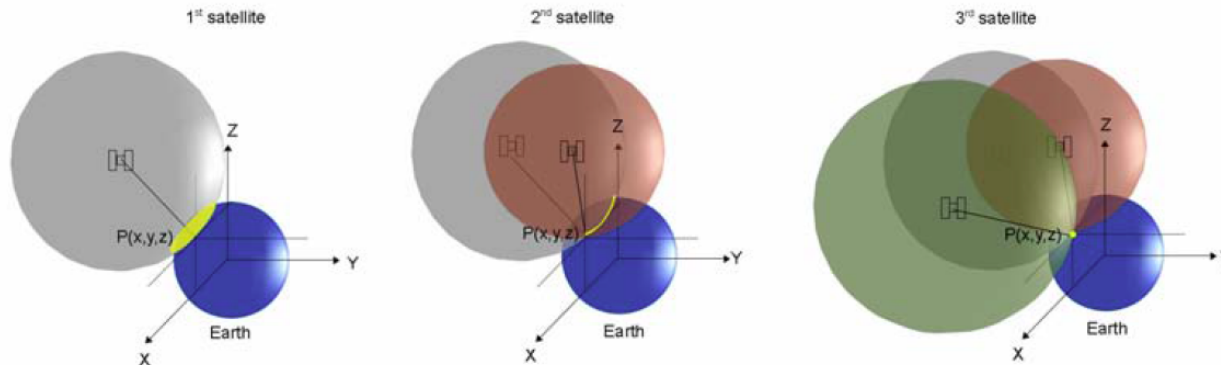
$$x[n] * \delta[n+s] = x[n+s]$$

Depending on the direction, this can be called a *delay* or an *advance*.

Common Impulse Responses

Science and engineering are filled with cases where one signal is a shifted version of another. E.g.

• GPS, radio signals time travel to measure distance

• In biology, the electrical signals in adjacent nerve cells are shifted versions of each other, as determined by the time it takes an action potential to cross the synaptic junction that connects the two.

Common Impulse Responses

The Next figure shows an impulse response composed of a delta function plus a shifted and scaled delta function.

By superposition, the output of this system is the input signal plus a delayed version of the input signal, i.e., **an echo**.

In this example, the echo is delayed by four samples and has an amplitude of 60% of the original signal.
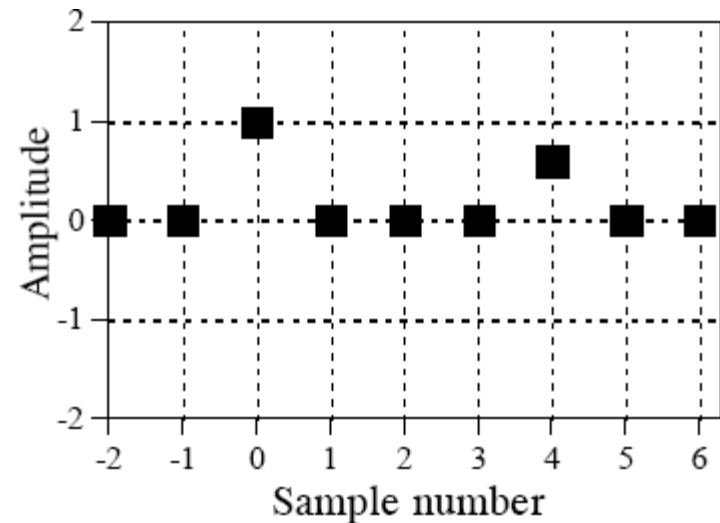
Common Impulse Responses

Echoes are important in many DSP applications.

The addition of echoes is a key part in making:
- Audio recordings sound natural and pleasant.
- Radar and sonar analyze echoes to detect aircraft and submarines.
- Geophysicists use echoes to find oil.
- Echoes are also very important in telephone networks, because you want to avoid them.

Calculus-like Operations

Calculus-like Operations

Convolution can change discrete signals in ways that resemble integration and differentiation.

Since the terms "derivative" and "integral" specifically refer to operations on continuous signals, other names are given to their discrete counterparts.

The discrete operation that mimics the **first derivative** is called the **first difference**.

Likewise, the discrete form of **the integral** is called the **running sum**.

Other names: discrete derivative and discrete integral,

Calculus-like Operations

In the **First Difference** each sample in the output signal is equal to the difference between two adjacent samples in the input signal.

$$y[n] = x[n] - x[n-1]$$

Each sample in the output signal is equal to the *difference* between adjacent samples in the input signal.

In other words, the output signal is the *slope* of the input signal.
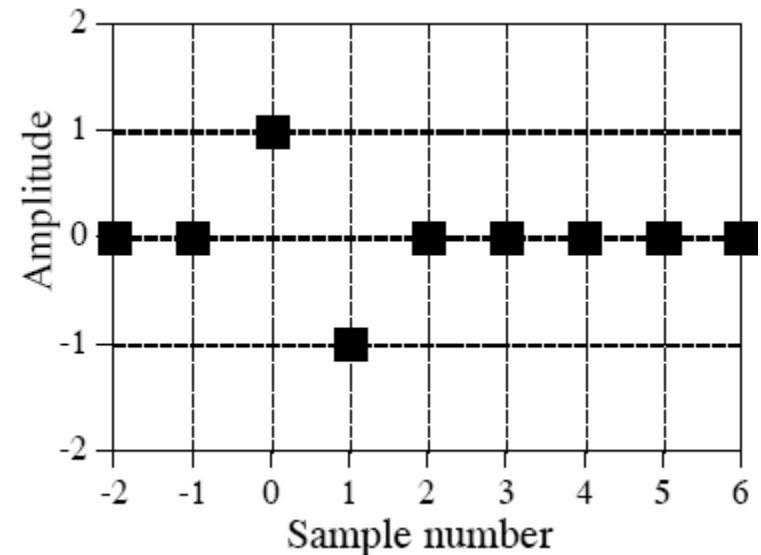
Calculus-like Operations

In **the Running Sum** each sample can be calculated by summing all points in the original signal to the left of the sample's location.

$$y[n] = x[n] + y[n-1]$$

Each sample in the output signal is equal to the sum of all samples in the input signal to the *left*.

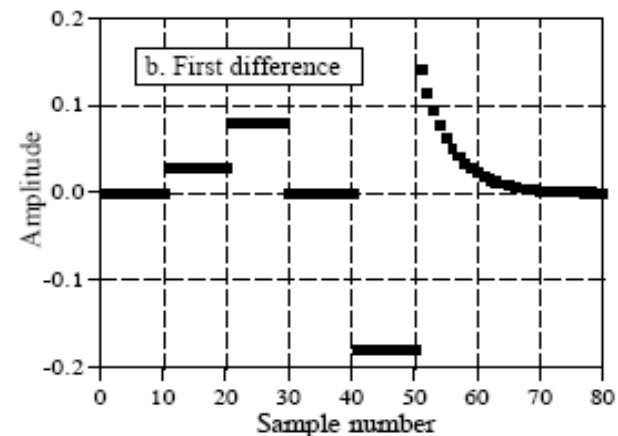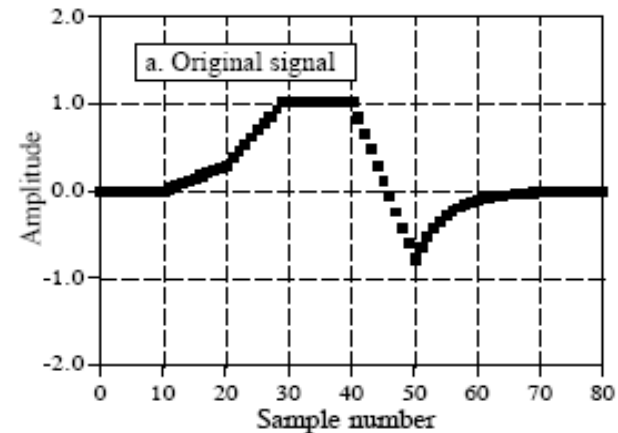Note that the impulse response extends to infinity.
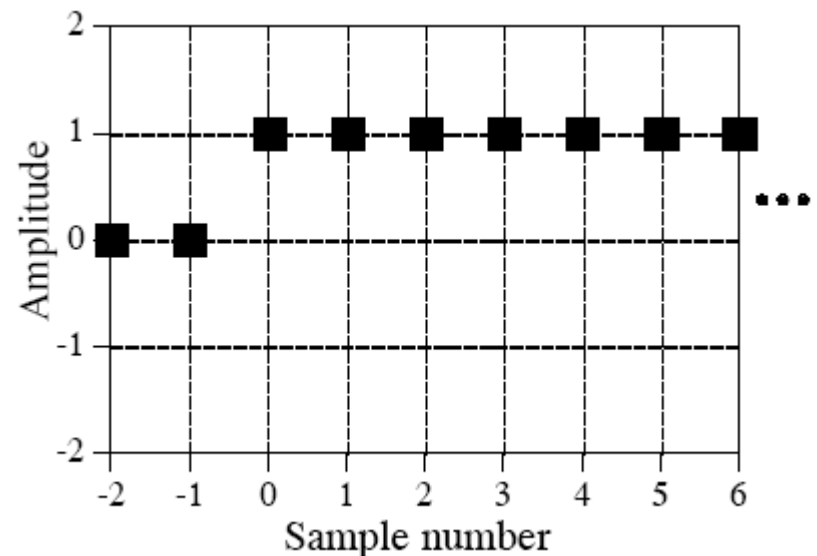
Calculus-like Operations

Example of calculus-like operations.

The signal in (b) is the first difference of the signal in (a).

Correspondingly, the signal in (a) is the running sum of the signal in (b).

a. Original signal

First Difference

Running Sum

b. First difference

Low-pass and High-pass Filters

Low-pass and High-pass Filters

Low-pass filter kernels are formed from a group of adjacent positive points that provide an averaging (smoothing) of the signal.
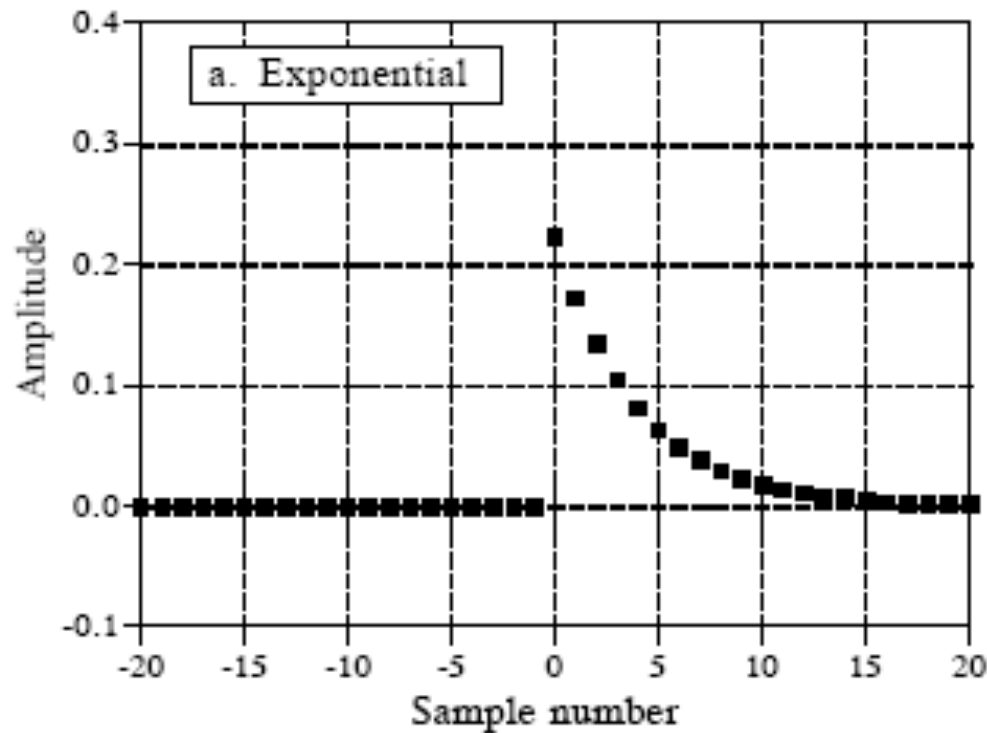
Each of these filter kernels is best for a particular purpose.

Just as in analog electronics, digital low-pass filters are used for:

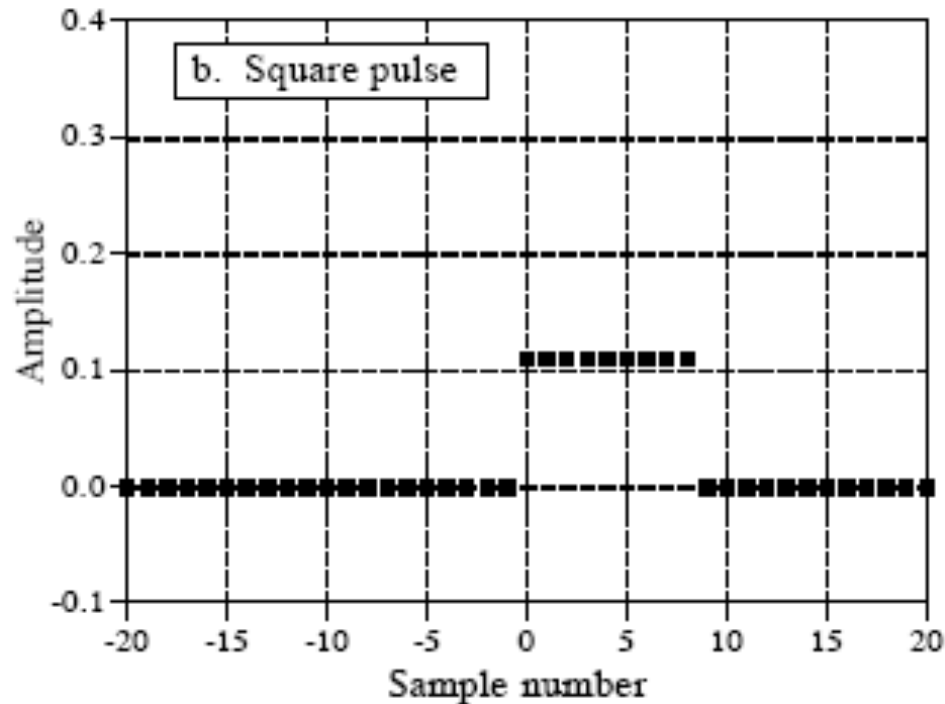   noise reduction
        signal separation
        wave shaping

Low-pass and High-pass Filters
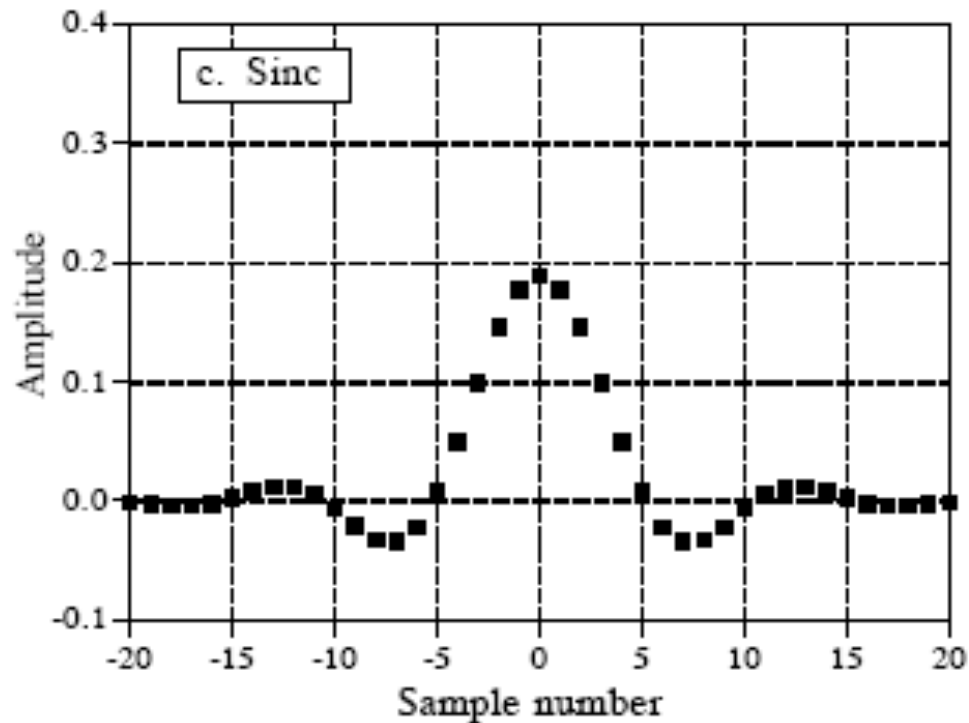
The exponential, is the simplest recursive filter.

Low-pass and High-pass Filters

The rectangular pulse, is best at reducing noise while maintaining edge sharpness.

b. Square pulse

Low-pass and High-pass Filters

The sinc function, a curve of the form: sin(x)/(x), is used to separate one band of frequencies from another.

Low-pass and High-pass Filters

The next figure shows three common high-pass filter kernels, derived from the corresponding low-pass filter kernels.

This is a common strategy in filter design:
First devise a low-pass filter and then transform it to what you need, high-pass, band-pass, band-reject, etc.

To understand the low-pass to high-pass transform, remember that a delta function impulse response passes the entire signal, while a low-pass impulse response passes only the low frequency components.

Low-pass and High-pass Filters

By superposition, a filter kernel consisting of a delta function minus the low-pass filter kernel will pass the entire signal minus the low- requency components resulting in a high-pass filter.

The next figures show typical high-pass filter kernels.

These are formed by subtracting the corresponding lowpass filter kernels from a delta function.

The distinguishing characteristic of high-pass filter kernels is a spike surrounded by many adjacent negative samples.

Low-pass and High-pass Filters
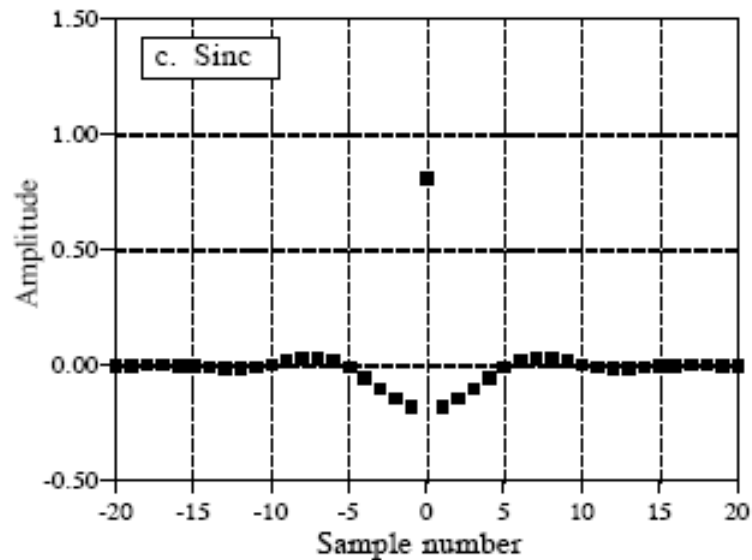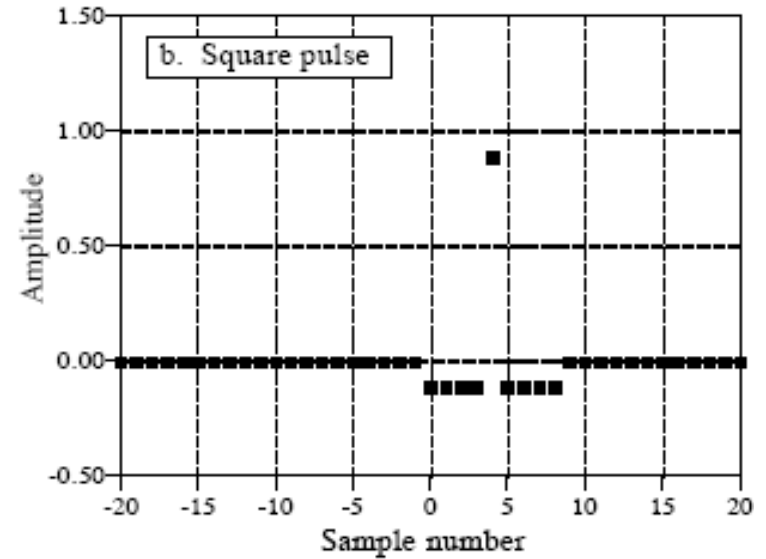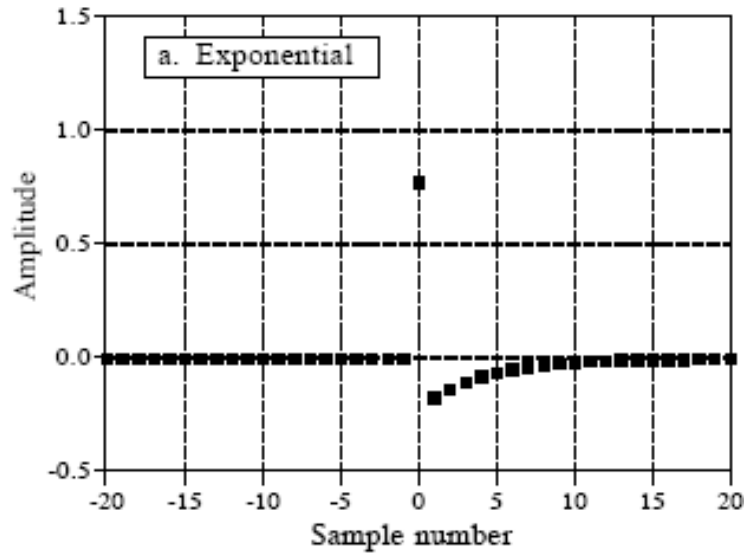
Causal and Noncausal Signals

Causal and Noncausal Signals

Imagine a simple analog electronic circuit. If you apply a short pulse to the input, you will see a response on the output.

Any effect must happen after the cause.

This is a basic characteristic of what we call ***time***.

Now compare this to a DSP system that changes an input signal into an output signal, both stored in arrays in a computer. If this mimics a real world system, it must follow the same principle of ***causality*** as the real world does.

Causal and Noncausal Signals

Systems that operate in this manner are said to be **causal**.

Since both the input and output signals are arrays of numbers stored in a computer, any of the input signal values can affect any of the output signal values.

As shown by the examples in the next figure, the impulse response of a causal system must have a value of zero for all negative numbered samples.

Causal and Noncausal Signals

In common usage, the term causal is applied to any signal where all negative numbered samples have a value of zero, whether it is an impulse response or not.
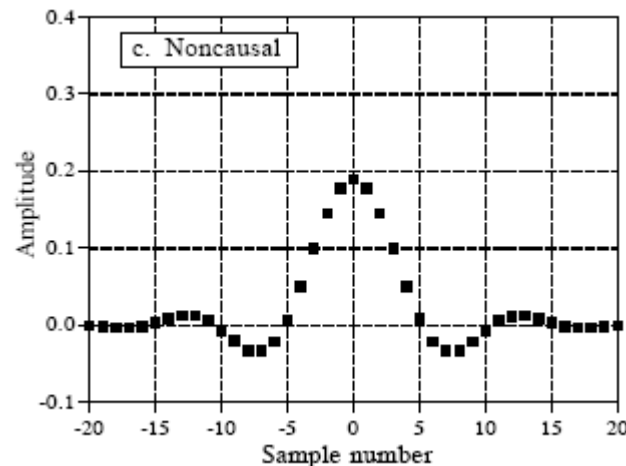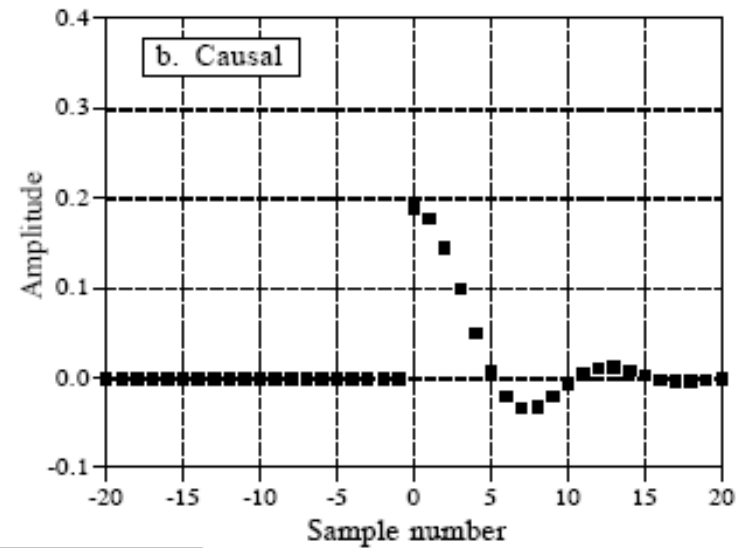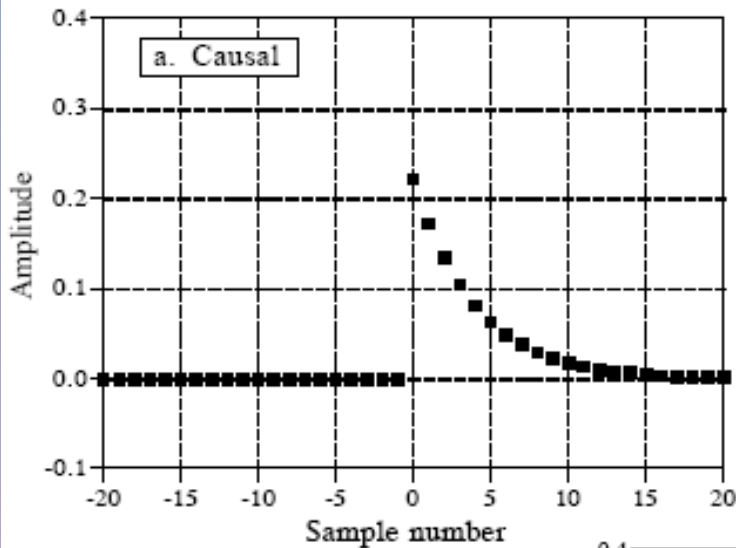
Zero Phase, Linear Phase, and Nonlinear Phase

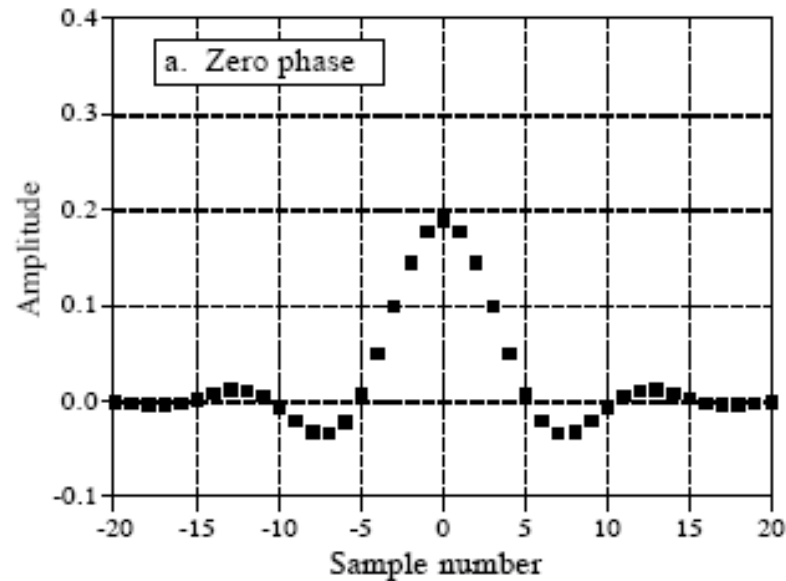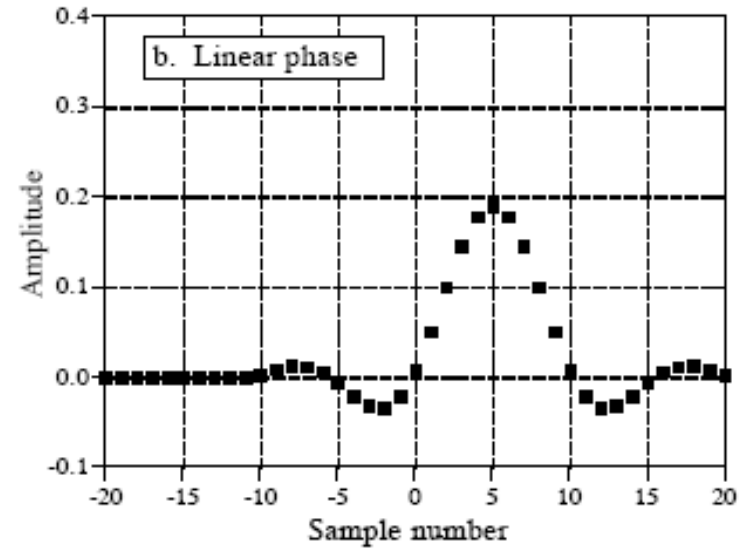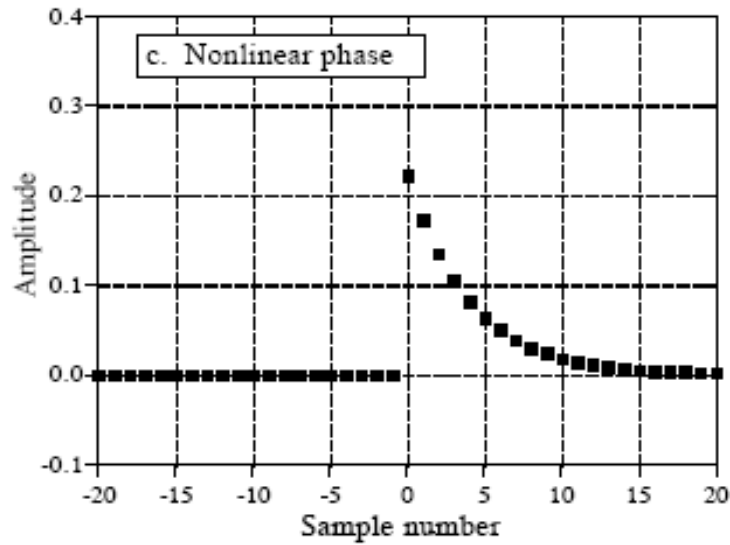# Zero Phase, Linear Phase, and Nonlinear Phase

Signals that have a left-right symmetry are said to be **linear phase**.

If the axis of symmetry occurs at sample number zero, they are additionally said to be **zero phase**.

Any linear phase signal can be transformed into a zero phase signal simply by shifting. Signals that do not have a leftright symmetry are said to be **nonlinear phase**.

Do not confuse these terms with the linear in linear systems. They are completely different concepts.

Zero Phase, Linear Phase, and Nonlinear Phase

Zero Phase, Linear Phase, and Nonlinear Phase

What does phase have to do with symmetry?

Briefly, the frequency spectrum of any signal is composed of two parts:

- *The magnitude.*

- *The phase*.

The frequency spectrum of a signal that is symmetrical around zero has a phase that is zero.

Zero Phase, Linear Phase, and Nonlinear Phase

Likewise, the frequency spectrum of a signal that is symmetrical around some nonzero point has a phase that is a straight line, i.e., a linear phase.

Lastly, the frequency spectrum of a signal that is not symmetrical has a phase that is not a straight line, i.e., it has a nonlinear phase.

Finaly, what does this have to do the concept of system linearity discussed in previously?
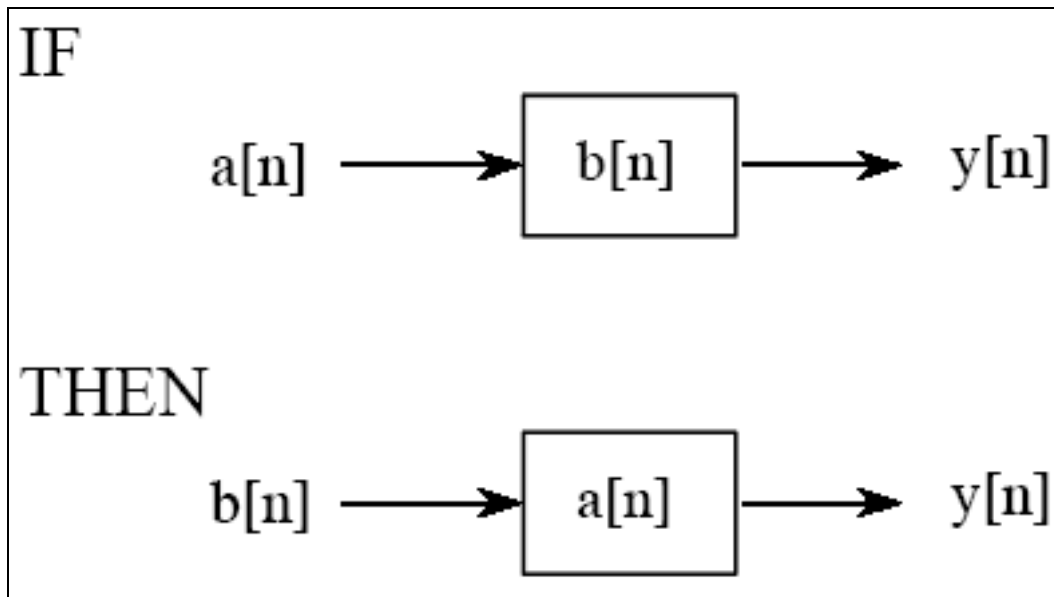
Absolutely nothing!

Mathematical Properties

Mathematical Properties

Commutative Property

This states that the order in which signals are convolved can be exchanged.
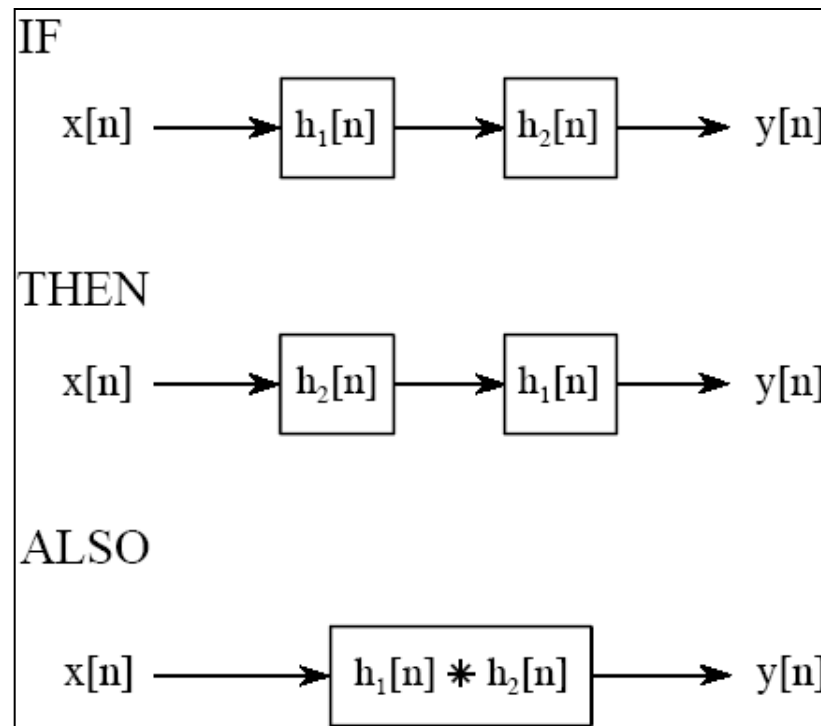
$$a[n] * b[n] = b[n] * a[n]$$

Mathematical Properties

Associative Property

The associative property of convolution describes how three or more signals are convolved.

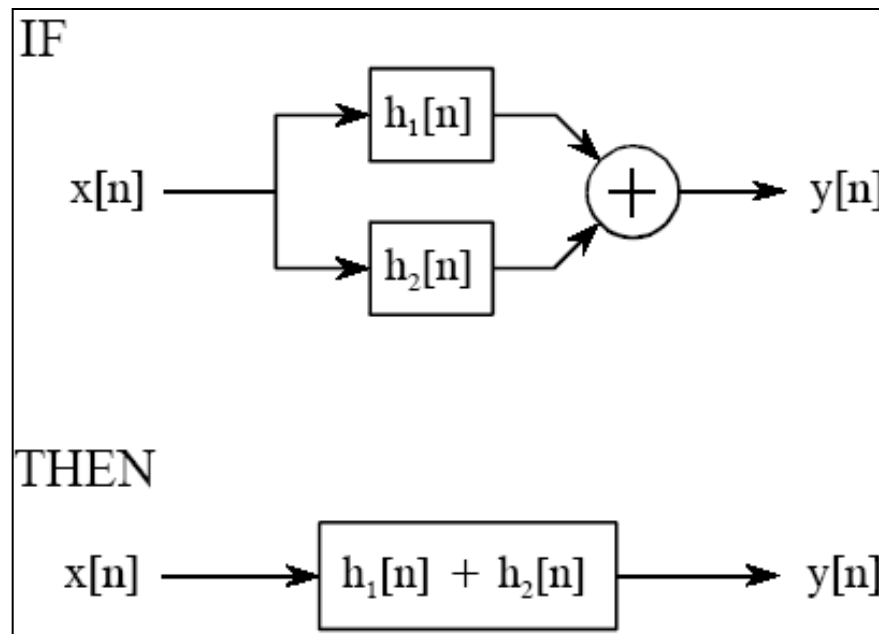$$(a[n] * b[n]) * c[n] = a[n] * (b[n] * c[n])$$

Mathematical Properties

Distributive Property

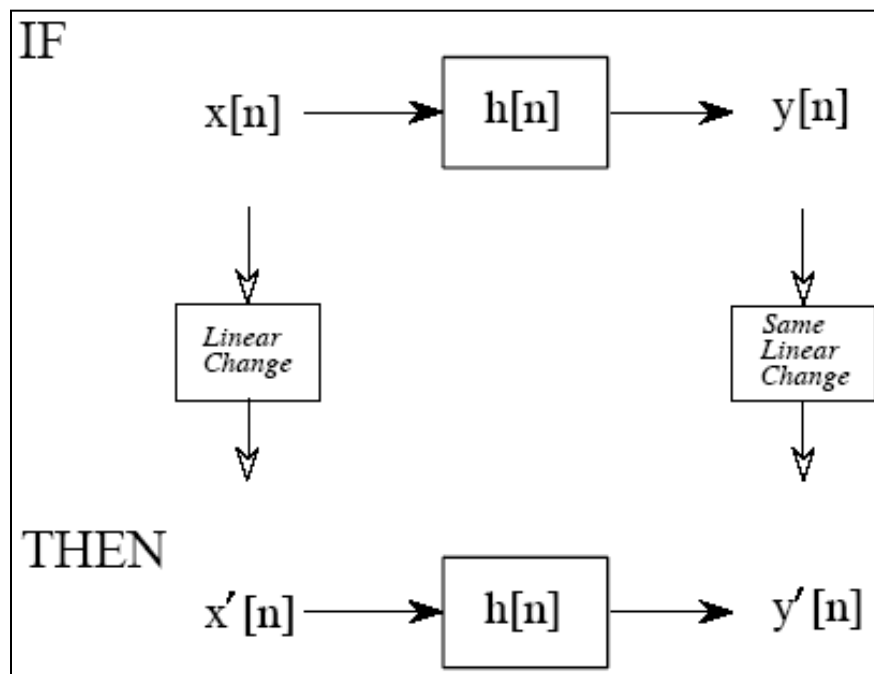This property of convolution describes how parallel systems are analyzed.

$$a[n] * b[n] \; + \; a[n] * c[n] \; = \; a[n] \; * \; (b[n] + c[n])$$

# Mathematical Properties

Transference between the Input and Output

This is a way of thinking about a common situation in signal processing. A linear change made to the input signal results in the same linear change being made to the output signal.
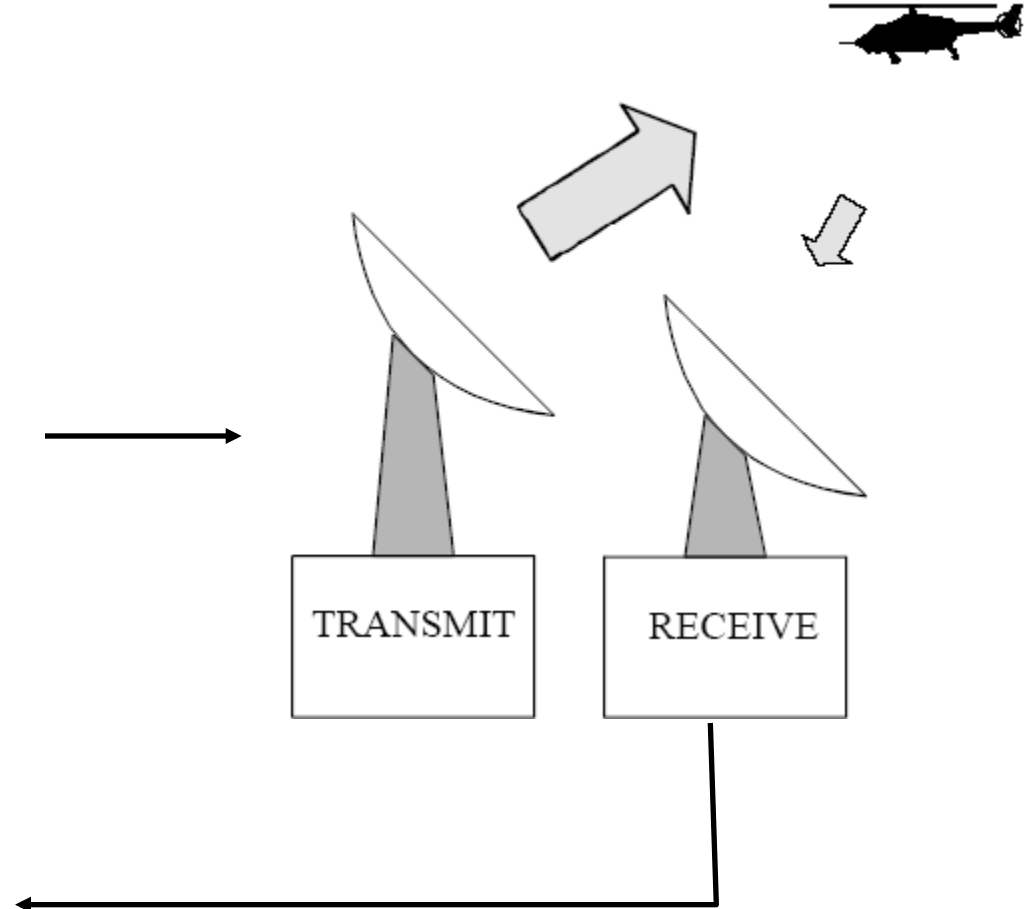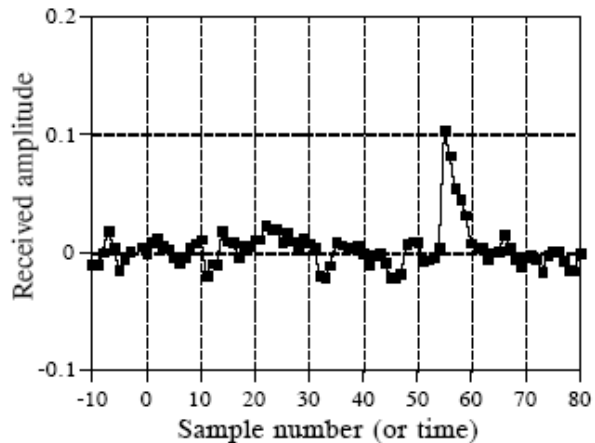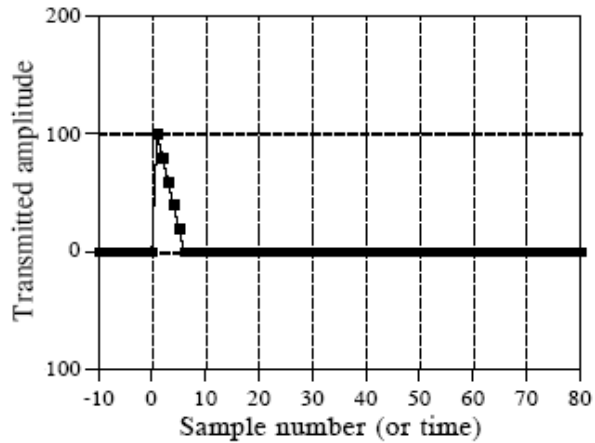
The Central Limit Theorem

- ## The Central Limit Theorem has an interesting implication for convolution.
  - If a pulse-like signal is convolved with *itself* many times, a Gaussian is produced.
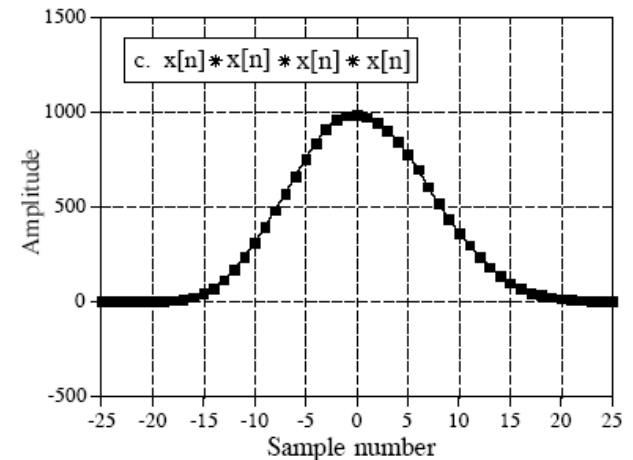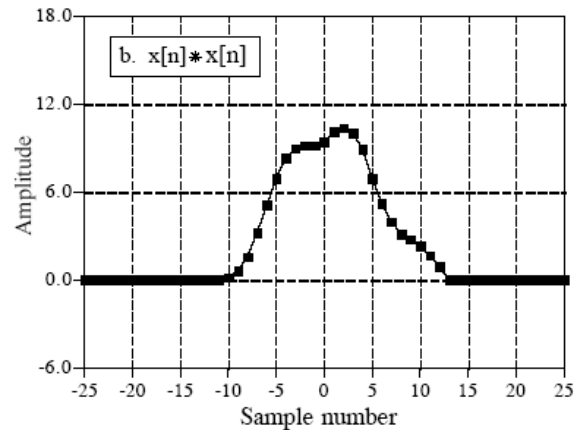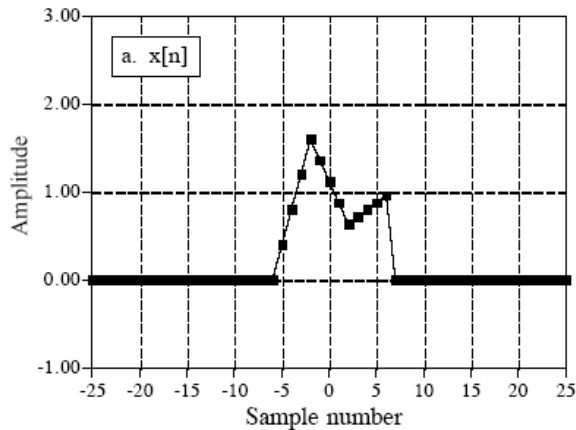
Correlation

Correlation

- A specially designed antenna transmits a short burst of radio wave energy in a selected direction.

- If the propagating wave strikes an object a small fraction of the energy is reflected back toward a radio receiver located near the transmitter.

- The transmitted pulse is a specific shape, the triangle in this example.

- The received signal will consist of two parts:
    - 1) a shifted and scaled version of the transmitted pulse, and
    - 2) random noise, resulting from interfering radio waves, thermal noise in the electronics, etc.

- Since radio signals travel at a known rate, the speed of light, the shift between the transmitted and received pulse is a direct measure of the distance

- This is the problem: given a signal of some known shape, what is the best way to determine where (or if) the signal occurs in *another* signal.

Correlation is the answer

Correlation

- Correlation is a mathematical operation that is very similar to convolution.

- Correlation uses two signals to produce a third signal.
  - This third signal is called the **cross-correlation** of the two input signals.

- If a signal is correlated with *itself*, the resulting signal is instead called the **autocorrelation**.

Correlation machine